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IN VORTEX RINGS THEORY***

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CONTENTS

RESUME :

Ce travail a pour objet l'étude numérique de certains aspects de la théorie des tourbillons stationnaires axisymétriques ("anneaux vortex" ou "doublets vortex plans") dans un fluide idéal. Après avoir rappelé l'origine physique des problèmes envisagés, nous indiquons divers résultats théoriques. Nous étudions ensuite un problème modèle analogue dans un domaine borné. La convergence d'une méthode d'éléments finis est établie dans un cadre général pour ce type de problèmes. Enfin, une méthode de "maillage à grandeur variable" est appliquée au problème des anneaux ou doublets vortex. On résout ainsi numériquement les problèmes "à vitesse de vortex libre" ou encore "à paramètre de flux libre". Les calculs mettent en évidence une relation entre ces deux paramètres qui s'interprète physiquement. Nous obtenons ainsi diverses configurations d'anneaux vortex en fonction des paramètres. Les calculs conduisent enfin à la formulation de diverses conjectures concernant ces problèmes.

ABSTRACT :

The aim of this paper is to study numerically some questions arising in the theory of axisymmetric vortex rings (or pairs) in an ideal fluid. We first recall the physical motivation for this problem and prove some theoretical results. We then study an analogous model problem in a bounded domain. The convergence of a finite element method is established in a general framework for this type of problem. Lastly, a variable mesh procedure is applied in the context of the vortex rings problem. This allows one to solve numerically the "free vortex velocity" problem and the "free flux parameter" problem. Computations exhibit a relationship between these two parameters which can be interpreted physically. They further lead to several conjectures for these problems.

KEY WORDS : Ideal fluid. Vortex rings. Vortex pairs. Velocity of the vortex. Flux parameter. Semi-linear elliptic problem in an unbounded domain. Finite element method. Iteration scheme. Numerical solution.

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1. INTRODUCTION.

In the description of axisymmetric steady vortex rings in an ideal fluid given in the classical work of Fraenkel and Berger [28], one is led to some free boundary problem for the Stokes stream function which reduces to a semi-linear elliptic problem in a half-plane[1]. With a view to making clear the physical significance of the various parameters involved in this problem and which play a crucial role, we start in the next section by recalling the derivation of the model. In Section 3, we show several qualitative properties of the solutions ; we also indicate some mathematical conjectures which arise quite naturally. Analogous problems but in a bounded domain are studied in Section 4 in a general framework. Existence results for the discrete versions of these model problems are proved in Section 5.

We then proceed in Section 6 to establish the convergence of the finite element approximation of these problems. Some algorithms used to solve the discrete problems are discussed in Section 7. Applications ot these results to the steady vortex rings or pairs problems are detailed in Sections 8 and 9. There, we require a variable mesh-size procedure in order to find suitable approximations of the problem in the unbounded domain.

Lastly, the numerical results are presented in Section 10. Several examples of vortex rings or pairs depending on the various parameters are shown together with the triangulations that were used. The computations appear to be quite efficient with respect to both the convergence speed and the accuracy (as can be checked by test identities). The computations give much evidence to support several conjectures that are outlined in Section 2 and which lead to some interesting mathematical questions. In particular, we exhibit a relationship between the velocity of the vortex ring and a certain flux parameter (see below) which lends itself to a physical interpretation.

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(1) An alternative approach to the question of steady vortex rings has been developopped by Benjamin [6]. It leads to related but different problems. The reader is referred to the works of Auchmuty and Benjamin [4] and Friedman and Turkington [29] for existence results in this direction.

2. PHYSICAL MOTIVATION OF THE PROBLEM.

In this Section, we recall the formulation of the problem of steady vortex rings in an ideal fluid given by Fraenkel and Berger [28]. We essentially follow here the elegant presentation that R. Temam [49, Appendix] has given for a problem in plasma physics and which is also valid here (see also Berestycki [8]).

Consider an ideal fluid, that is, inviscid and incompressible, which is in axisymmetric equilibrium around an axis Oz. Let $\vec{v}$ denote the velocity field of the fluid and $\vec{\omega} = \vec{\text{curl}}\ \vec{v}$ the vorticity field. A vortex ring is defined as an axisymmetric region $\mathcal{A}$ of the space $\mathbb{R}^3$ around Oz such that $\vec{\omega} \neq \vec{0}$ in $\mathcal{A}$ while $\vec{\omega}=\vec{0}$ outside $\mathcal{A}$. From the equation of mass conservation

$$\text{(2.1)} \qquad \operatorname{div} \vec{v} = 0,$$

and using the axial symmetry, one derives the existence of the Stokes stream function $\psi = \psi(r,z)$ for the fluid (where (r,θ,z) are the usual cylindrical coordinates in $\mathbb{R}^3$). That is, letting (v_r, v_θ, v_z) denote the coordinates of $\vec{v}$ in the cylindrical basis, one has :

$$\text{(2.2)} \qquad v_r = \frac{1}{r}\frac{\partial\psi}{\partial z} \qquad v_z = -\frac{1}{r}\frac{\partial\psi}{\partial r} .$$

(Recall that $\vec{v}$ only depends on r and z). Let Π be the half-plane

$$\Pi = \{(r,z) \in \mathbb{R}^2 \ ; \ r > 0\}$$

and let A be the cross section of $\mathcal{A}$ in the meridian half-plane Π (i.e. θ = constant).

In the cylindrical basis, the vorticity $\vec{\omega} = \vec{\text{curl}}\ \vec{v}$ has the following expression :

$$\text{(2.3)} \qquad \vec{\omega} = \left(-\frac{1}{r}\frac{\partial\sigma}{\partial z}, -\mathcal{L}\psi, \frac{1}{r}\frac{\partial\sigma}{\partial r}\right),$$

where $\sigma = rv_\theta$ and $\mathcal{L}$ is the following self-adjoint elliptic operator :

$$\mathcal{L}\psi = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) - \frac{1}{r}\frac{\partial^2\psi}{\partial z^2} .$$

Thus, from (2.3) it follows that

(2.4) $\mathcal{L}\psi = 0$ in π-A.

The field being ideal and stationnary, the velocity field also verifies the Euler equation

(2.5) $$(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p$$

where p is the pressure and may contain gravitational terms (assuming $\vec{g}$ is directed along Oz). More generally, the term p may also involve potentials for other force fields, provided they are axisymmetric. Recalling the relation (valid for $\vec{\omega}$ = curl $\vec{v}$)

(2.6) $$\vec{\omega}\times\vec{v} + \frac{1}{2}\vec{\nabla}|\vec{v}|^2 = (\vec{v}\cdot\vec{\nabla})\vec{v}$$

one derives

(2.7) $$\vec{\omega}\times\vec{v} = -\vec{\nabla}F \ , \ F = p + \frac{1}{2}|\vec{v}|^2.$$

Then, using (2.2),(2.3) and aximymmetry, (2.7) yields :

(2.8) $$\begin{cases} -\dfrac{\partial F}{\partial r} = \dfrac{1}{r}\mathcal{L}\psi\,\dfrac{\partial\psi}{\partial r} - \dfrac{1}{2r^2}\dfrac{\partial\sigma^2}{\partial r} \\ \dfrac{\partial F}{\partial\theta} = 0 = -\dfrac{1}{r^2}\left\{\dfrac{\partial\sigma}{\partial r}\dfrac{\partial\psi}{\partial z} - \dfrac{\partial\sigma}{\partial z}\dfrac{\partial\psi}{\partial r}\right\} \\ -\dfrac{\partial F}{\partial z} = \dfrac{1}{r}\mathcal{L}\psi\,\dfrac{\partial\psi}{\partial z} - \dfrac{1}{2r^2}\dfrac{\partial\sigma^2}{\partial z}\ . \end{cases}$$

From the second equation in (2.8), one sees that $\vec{\nabla}\sigma$ and $\vec{\nabla}\psi$ always have the same direction. Hence, σ and ψ have the same level surfaces : σ is a function of ψ and one sets $\sigma^2 = f_o(\psi)$. Thus, $\vec{\nabla}(\sigma^2) = f'_o(\psi)\vec{\nabla}\psi$. Then, (2.8) yields :

(2.9) $$-\vec{\nabla}F = \left(\frac{1}{r}\mathcal{L}\psi - \frac{1}{2r^2}f'_o(\psi)\right)\vec{\nabla}\psi.$$

Whence, again, F is a function of ψ and one sets $F = f_1(\psi)$. This leads to the equation

$$(2.10) \qquad \mathcal{L}\psi = f(r,\psi) \text{ in } A,$$

where $f(r,\psi) = -rf_1'(\psi) + \frac{1}{2r} f_0'(\psi)$. This function is called the <u>vorticity function</u>. Indeed, recall that one has

$$\omega_\theta = -f(r,\psi).$$

(Compare (2.3) and (2.10)). We will always consider here that this function f <u>is given</u>, an assumption which seems reasonable from the viewpoints of applications.

For physical reasons, one further assumes that

$$(2.11) \qquad \vec{v} \text{ is continuous across } \partial A$$

$$(2.12) \qquad \vec{v}\cdot\vec{n} = 0 \text{ on } \partial A$$

where $\vec{n}$ is the outward unit normal vector on ∂A. This last condition just says that $\partial\mathcal{A}$ is a stream surface. By (2.11), ψ is C^1 across ∂A and, since $\vec{v}\cdot\vec{n} = -\frac{\partial\psi}{\partial\vec{\tau}}$, where $\vec{\tau}$ is the unit tangent vector on ∂A, one derives from (2.12) that ψ is constant on ∂A. ψ being defined up to the addition of a constant we set

$$(2.13) \qquad \psi = 0 \text{ on } \partial A.$$

By axisymmetry, the axis $Oz = \partial\Pi$ has to be a streamline. Therefore,

$$(2.14) \qquad \psi = -k \text{ on } \partial\Pi \quad (k \text{ being a constant}).$$

Notice that k is a <u>flux parameter</u> : $2\pi k$ represents the flux rate between the boundary ∂A and the axis Oz.

One makes the assumption that the <u>fluid is at rest at infinity</u> and that the vortex ring moves along the axis Oz at a <u>constant velocity</u> $-W < 0$. For our formulation, it is equivalent and more convenient to consider the ring as being fixed and that the fluid at infinity has uniform velocity W in the direction Oz. That is,

(2.15) $\quad \vec{v} \sim (0,0,W)$ as $(r,z) \to \infty$.

We nonetheless continue to regard W as the "vortex velocity". Hence, at infinity, the stream function satisfies the "boundary condition" :

(2.16) $\quad \psi \sim \psi_\infty$ and $\nabla\psi \sim \nabla\psi_\infty$ as $r+|z| \to \infty$, with $\psi_\infty = -\frac{W}{2}r^2-k$.

Let $u = \psi + \frac{W}{2}r^2 + k$; u represents the perturbation of the stream function due to the vorticity motion. We interpret (2.16) by requiring

(2.17) $\quad u \in L^2(\Pi)$, $\frac{1}{r}|\nabla u|^2 \in L^1(\Pi)$.

Lastly, as a normalization condition which is quite natural in the present context, one prescribes the value $\eta > 0$ of the kinetic energy of the vorticity motion. That is (*) :

(2.18) $$\int_\Pi \frac{1}{r}|\nabla u|^2 \, dx = \eta \quad .$$

The problem is now completely formulated in the equations (2.4), (2.10)-(2.14), (2.17) and (2.18). It is primarily set as a free boundary problem. The vorticity region A is a priori unknown. Together with the stream function u and one free parameter (either W or k or else a "vortex strength" parameter) one seeks to determine ∂A from the parameters which are prescribed. Following Fraenkel and Berger [28], this free boundary problem can however be reduced to a semilinear elliptic problem. To this end, we further impose the physically natural restriction :

(2.19) $\quad f(r,\psi) > 0$ in A.

By the maximum principle, it is then straightforward to see that $A = \{x \in \Pi \ ; \ \psi(x) > 0\}$ while $\Pi \setminus A = \{x \in \Pi \ ; \ \psi(x) < 0\}$.

(*) In all the sequel dx represents the Lebesgue measure integration element on Π.

Therefore, we shall assume

(2.20) $\qquad f(r,z) > 0 \quad \forall z > 0,$

and we extend f by setting

(2.21) $\qquad f(r,z) = 0 \quad \forall z \leq 0.$

Thus, the problem (2.4), (2.10)-(2.14), (2.17) and (2.18) is now equivalent to :

<u>Problem</u> (I) (three-dimensional)

$$\text{(I)} \qquad \begin{cases} \mathcal{L}u = f(r, u - \frac{W}{2} r^2 - k) \text{ in } \Pi, \\ u = 0 \text{ on } \partial\Pi,\ u \in L^2(\Pi),\ \frac{1}{r} |\nabla u|^2 \in L^1(\Pi),\ u \in C^1(\Pi), \\ \int_\Pi \frac{1}{r} |\nabla u|^2 \, dx = \eta. \end{cases}$$

In the case of a two-dimensional fluid, in axisymmetric equilibrium, one obtains an analogous problem describing the existence of vortex pairs (see e.g. J. NORBURY [40]) :

<u>Problem</u> (II) (two-dimensional)

$$\text{(II)} \qquad \begin{cases} -\Delta u = f(u - Wr - k) \text{ in } \Pi \\ u \in H_o^1(\Pi),\ u \in C^1(\Pi) \\ \int_\Pi |\nabla u|^2 \, dx = \eta \end{cases}$$

Problems (I) and (II) serve as models for equilibrium in various related phenomena that arise in other contexts. In ideal fluid mechanics, as we just have seen, they model the existence of vortex rings or pairs. (A typical example of a vortex ring is a smoke ring). They also arise in meteorology, physics of low temperatures, supraconductivity, quantum theory of suprafluid helium etc...

Related problems but leading to somewhat different mathematical difficulties arise in several other contexts :

- In plasma physics, for the "Grad-Shafranov" equations of equilibrium for a plasma confined in a Tokamak machine ; see R. Temam [49,50], H. Berestycki and H. Brezis [10,11], J.P. Puel [44] and M. Sermange [48].
- In astrophysics, as models for self-gravitating stars ; see J.F.G. Auchmuty and R. Beals [3] and the recent and more general work of P.L. Lions [37,38].
- In Thomas-Fermi theory of atoms ; see E. Lieb and B. Simon [36], P. Bénilan and H. Brézis [5], H. Brézis [17], H. Brézis, R. Benguria and E. Lieb [18].
- In related steady inviscid flows with vorticity ; see J.P. Christiansen and N.J. Zabusky [19], G.S. Deem and N.J. Zabusky [23], L. Lichtenstein [33,34], R.J. Pierrehumbert [43], P.G. Saffman [45,46], P.G. Saffman and J.C. Schatzman [47].
- In internal waves in inviscid stratified flows ; see T.B. Benjamin [6].

3. SOME THEORETICAL RESULTS AND OPEN PROBLEMS.

Let us now state more precisely the various problems we want to consider in the framework of the systems (I) and (II). We also show here several theoretical results concerning these problems. Along the way, we will have the occasion to formulate some mathematical conjectures which seem quite natural and which are strongly supported by the numerical evidence presented in the last section.

We shall always assume thereafter that the vorticity functions satisfy the following set of hypotheses (f is either of the form $f = f(r,s)$ as in (I) or of the form $f = f(s)$ as in (II)).

$$\text{(f.1)} \qquad \begin{cases} f : \mathbb{R}_+ \times \mathbb{R} \to [0,+\infty) \text{ is locally Hölder continuous ;} \\ f(r,s) = 0 \quad \forall s \le 0, \text{ and } f(r,s) > 0 \quad \forall s > 0. \end{cases}$$

$$\text{(f.2)} \qquad f(r,s) \le f(r,s') \ , \quad \forall s \le s' \quad \forall r \in \mathbb{R}_+$$

$$\text{(f.3)} \qquad 0 \le f(r,s) \le as^p + b \quad \forall s \ge 0, \quad \forall r \in \mathbb{R}_+$$

with $a,b > 0$ and $p > 1$ being constants, with no restriction imposed on p.

(f.4) $\lim_{s \to +\infty} f(r,s) = +\infty$, uniformly on compact sets of $r \in \mathbb{R}_+$.

3.1. Variational formulations and results.

Let us first recall that Fraenkel and Berger [28] and Norbury [40] have studied a nonlinear eigenvalue problem associated with (I) or (II) :

Problem (I) - λ : Given $\eta > 0$, $W > 0$ and $k \geq 0$, to find u and $\lambda > 0$ satisfying

$$\text{(I)} \quad \begin{cases} \mathcal{L}u = \lambda f(r, u - \frac{W}{2} r^2 - k) \text{ in } \Pi, \\ u \in L^2(\Pi), \ \frac{1}{r} |\nabla u|^2 \in L^1(\Pi), \ u=0 \text{ on } \partial\Pi, \\ \int_\Pi \frac{1}{r} |\nabla u|^2 \, dx = \eta . \end{cases}$$

Problem (II)- λ : Given $\eta > 0$, $W > 0$ and $k \geq 0$, to find $u \in H_0^1(\Pi)$ and $\lambda > 0$ satisfying

$$\text{(II)} \quad \begin{cases} -\Delta u = \lambda f(u - Wr - k) \text{ in } \Pi, \ u \in H_0^1(\Pi) \\ \int_\Pi |\nabla u|^2 \, dx = \eta . \end{cases}$$

In these formulations, λ can be interpreted as a "vortex strength" parameter which is a priori unknown. Relying on an approximation of Π by bounded domains, and using a variational formulation for the analogous problems in the bounded domains, Fraenkel and Berger [28] have shown that if f satisfies essentially (f.1)-(f.3), then problem (I)-λ admits a solution u,λ. Norbury [40] has obtained a similar result for (II)-λ. Using a direct and simpler variational approach, these results have recently been extended by H. Berestycki and P.L. Lions [12], under more general hypotheses on f than (f.1)-(f.3)(*). In the case of (II)-λ, for instance, the variational problem considered in [40] reads :

(*) Note however that [12] only deals with continuous vorticity functions f. When f possesses a discontinuity (f is a Heaviside function), the results are obtained in [28] by a limiting procedure.

$$\begin{cases} \text{maximize } \int_{\Pi} F(u-Wr-k) \text{ on the constraint set} \\ u \in H^1_o(\Pi) \text{ and } \int_{\Pi} |\nabla u|^2 = \eta, \end{cases}$$

where $F(z) = \int_0^z f(s)ds$. Let us observe that because of this variational formulation, the results of H. Berestycki and C. Stuart [13,14] can be applied to show the global convergence of certain algorithms that one uses for the numerical solution of these nonlinear eigenvalue problems (see [13,14] for more details in this direction).

One will find a different variational formulation of vortex ring problems in T.B. Benjamin [6]. The reader is also referred to J.F.G. Auchmuty and R. Beals [3] and A. Friedman and B. Turckington [29].

Another approach to the problem of vortex rings concerns the existence of nontrivial solutions of the first two equations in (I) or (II). This is the "free kinetic energy problem" since then $\int_{\Pi} \frac{1}{r} |\nabla u|^2 dx$ or $\int_{\Pi} |\nabla u|^2 dx$ are not prescribed any longer : Given $W > 0$ and $k \geq 0$, one looks for a solution $u > 0$ and a number $\eta > 0$ satisfying (I) or (II). Existence results from this standpoint have recently been given independently by Ambrosetti and Mancini [1,2] and W.M. Ni [39] under (f.1)-(f.4) and some additional assumptions on f. These works both rely on variational techniques and an approximation of Π by bounded domains.

3.2. Formulations involving other parameters of the problem.

From a physical point of view though it seems also interesting to consider the case where the vorticity function is completely given (hence λ is fixed, say $\lambda=1$, in (I)-λ or (II)-λ) and the kinetic energy of the vorticity motion η is a priori prescribed. Then, one of the parameters W or k is given and the other one is free. One is thus led to the following problems :

Problem (I)-W (resp. (II)-W) : "Free vortex velocity problem" :
Given $\eta > 0$, $k \geq 0$, to find u and $W > 0$ satisfying (I) (resp. (II)).

Problem (I)-k (resp. (II)-k) : "Free flux parameter problem" :
Given $\eta > 0$, $W > 0$, to find u and $k \geq 0$ satisfying (I) (resp. (II)).

Throughout this paper, we will be mainly concerned with these two problems. Let us first observe that, on the contrary to all the others mentioné before, these two problems do not seem to have a variational structure. That is, one cannot - at least obviously - find a solution of e.g. (I)-W by obtaining u as a critical point of some functional and W as a Lagrange multiplier associated with a certain constraint. This feature of the problem combines with the fact that Π is unbounded to make existence results harder to achieve from a mathematical viewpoint in the framework of (I)-W, (I)-k or (II)-k. Whence, we believe, the special interest in the numerical computations for these problems, the phenomena they display and the conjectures to which they lead.

3.3. Some theoretical results and conjectures.

Let us now recall some theoretical results from H. Berestycki [9] (see also H. Berestycki- P.L. Lions [12]). Firstly we remark that because of its special features, problem (II)-W is easily solvable via problem (II)-λ.

Theorem 1 : Assume f satisfies (f.1)-(f.4) and let $\eta > 0$ and $k \geq 0$ be given. Then, there exists a solution $u \in H^1_0(\Pi)$ and $W > 0$ of the free vortex velocity problem (II)-W.

Proof : By J. Norbury [40] (see also H. Berestycki and P.L. Lions [11] with some more general hypotheses), we know that there exists a solution $u \in H^1_0(\Pi)$ and $\lambda > 0$ of problem (II)-λ, where we take W=1 :

$$\begin{cases} -\Delta u = \lambda f(u-r-k) \text{ in } \Pi \ ; \\ u \in H^1_0(\Pi) \ ; \ \int_\Pi |\nabla u|^2 = \eta \ . \end{cases}$$

Then, one can operate a scale change

$$v(r,z) = u(r/\sqrt{\lambda}\ , z/\sqrt{\lambda}\ )$$

and v satisfies

$$\begin{cases} -\Delta v = f(v-Wr-k) \text{ in } \Pi \\ v \in H^1_0(\Pi) \\ \int_\Pi |\nabla v|^2 \, dx = \int_\Pi |\nabla u|^2 \, dx = \eta \end{cases}$$

with $W = \frac{1}{\sqrt{\lambda}}$. Therefore, (v,W) is a solution of (II)-W. ∎

In the three dimensional problem (I)-W, however, this scale change does not preserve the constraint any longer. But we nevertheless conjecture that the result remains true in this case :

Conjecture 1 : For any $\eta > 0$, $k \geq 0$, problem (I)-W possesses at least one solution.

For the free flux parameter problems, we collect the following qualitative results :

Theorem 2 : Assume f satisfies (f.1)-(f.4) and let $\eta > 0$ be given, then there exists $W_1^* > 0$ (resp. $W_2^* > 0$) such that for any $W > W_1^*$ (resp. $W > W_2^*$), problem (I)-k (resp. (II)-k), does not admit any solution.

Remark 3.1 : The above results says that once the kinetic energy is fixed, the velocity of the vortex is a priori limited, independently of k. This property is indeed physically intuitive. ∎

Theorem 3 : Under conditions (f.1)-(f.4), $\eta > 0$ being given for any $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that for $k \geq K_\varepsilon$, any solution (u,W) of (I)-W or of (II)-W verifies $W < \varepsilon$.

Remark 3.2 : In informal terms, Theorem 3 means that once the kinetic energy is fixed, one has the following asymptotic relationship between W and k : "$k \to +\infty \Longrightarrow W \to 0$". ∎

For the detailed proofs, the reader is referred to H. Berestycki [9]. We just outline here the idea of the argument. For the sake of simplicity, we only consider the case of problem (II).

Sketch of the proof of Theorem 2.

Step 1 : Symmetry property. Let $W > 0$ be given. Using the results of Gidas, Ni and Nirenberg [30] (and more precisely, in this case, of M.J. Esteban [24], a solution (u,k) of problem (II)-k is necessarily Steiner symmetric with respect to an axis parallel to $\vec{Or}$. After a translation along the direction $\vec{Oz}$ - notice that the problem is invariant under such translations - one may assume that u is Steiner symmetric with respect to the axis $\vec{Or}$. That is, u verifies :

(3.1) $\quad u(r,-z) = u(r,z) > 0$

(3.2) $\quad z \mapsto u(r,z)$ is decreasing for $z \in [0,+\infty)$.

Step 2 : A priori estimates on the vorticity region. Let $W_* > 0$ be fixed and let (u,k) be a solution of problem (II)-k corresponding to some $W \geq W_*$. Using a method of Fraenkel and Berger [28] and Norbury [40] that relies on precise and sharp inequalities for Green's function in Π, one can show (see [8]) that, because of (3.1)-(3.2), the vorticity region

$$\Omega_+ = \{(r,z) \in \Pi \ ; \ u(r,z) > Wr+k\}$$

verifies

(3.3) $$\Omega_+ \subset \{(r,z) \in \Pi \ ; \ 0 < r < r_*, \ |z| < z_*, \ |z| < \frac{C}{(Wr+k)^2}\},$$

where r_* and z_* are positive constants which only depend on W_*.

Step 3 : L^∞ estimate on u. Using (3.3) and the integral formulation of (II) by means of Green's function, one can show that u satisfies the estimate

(3.4) $$\|u\|_{L^\infty(\Pi)} \leq \frac{C(\eta)}{W}$$

where $C(\eta)$ is a positive constant which only depends on W_*.

Step 4 : Conclusion. Integration by parts in (II) yields :

(3.5) $$\eta = \int_\Pi |\nabla u|^2 \, dx = \int_\Pi f(u-Wr-k)u \, dx.$$

Whence, using (f.1) :

(3.6) $$\eta = \int_{\Omega_+} f(u-Wr-k)u \, dx.$$

It is then straightforward to derive from (3.6) using (3.4) that

(3.7) $$\eta \leq \frac{C}{W}$$

where $C > 0$ is given and only depends on η and W_*.

Therefore, there exists $W^* > 0$ ($W^* = W^*(\eta)$) such that for any $W > W^*$, (3.7) is impossible and thus, (II)-k has no solution. ■

Sketch of the proof of Theorem 3. We argue indirectly. Assume there exists a sequence u_n, W_n, k_n of solution to (II) such that $k_n \to +\infty$ while $W_n \geq W_*$ for some $W_* > 0$. By (3.3) and (3.7), we know that $\Omega_{n+} = \{x \in \Pi \ ; \ u_n(x) \geq W_n r + k_n\}$ remains bounded and $\|u_n\|_{L^\infty} \leq C$ for some constant $C > 0$. Therefore, it follows from (3.6) that

$$\eta \leq C \text{ meas } (\Omega_{n+}), \tag{3.8}$$

where meas (Ω_{n+}) denotes the Lebesgue measure of Ω_{n+}, and $C > 0$ is a constant. On the other hand, the estimate (3.3) shows that meas $(\Omega_{n+}) \to 0$ as $n \to +\infty$ since $k_n \to +\infty$ and $W_n \geq W_* > 0$. Therefore (3.8) implies that such a sequence cannot exist. Alternatively, (3.3) and (3.8) show that for any $W_* > 0$, there exists $K(W_*) > 0$ such that any solution (u,k) of (II)-k corresponding to $W \geq W_*$ must verify $k \leq K(W_*)$, which is the same statement as Theorem 3. ■

It is quite tempting to complete the qualitative description given in the preceding results by some conjectures.

Conjecture 2 : For any W, $0 < W \leq W_1^*$ (resp. $0 < W < W_2^*$), there exists a solution (u,k) of (I)-k (resp. (II)-k).

Conjecture 3 : Let $\eta > 0$ be fixed. For any $K > 0$, there exists $\varepsilon(K) > 0$ such that any solution (u,W) of (I)-W or (II)-W corresponding to $k \geq K$ verifies $W \leq \varepsilon(K)$.

Remark 3.3 : This last statement is a converse to Theorem 3. It means "$W \to 0 \Longrightarrow k \to +\infty$". ■

Symmetry properties lead us to formulate our last

Conjecture 4 : W_1^* (resp. W_2^*) is given by the solution to problem (I)-W (resp. (II)-W) in the case k=0.

Remark 3.4 : For the three-dimensional problem (I)-W, in the case f is a Heaviside function $f(r,s) = r\, h(s)$, with $h(s) = 1$ if $s > 0$ and $h(s) = 0$ if $s \leq 0$, then there is a solution of (I)-W with k=0 known as Hill's spherical vortex (see Fraenkel and Berger [23]). In this case Ω_+ is a half-circle. For the two-dimensional problem (II)-W, the same property holds when $f(x) = \lambda s^+$ with $\lambda > 0$ being a constant and $s^+ = \max\{s,0\}$. In this case again, there is a solution of (II)-W corresponding to k=0, such that Ω_+ is a half circle. This solution is known as Hill's cylindrical vortex. It would be interesting to see, in view of the preceding conjecture, whether one can characterize this case by a kind of "isoperimetric inequality". ∎

It will be seen in Section 10 that the above conjectures are strongly supported by numerical evidence. From a purely mathematical viewpoint though, they seem completely open at the moment.

3.4. An integral identity.

For the solutions of vortex ring problems some integral identities can be derived which will further provide the possibility of useful checks on the numerical work (see Section 10 below). These identities were first reported in M.S. Berger and L.E. Fraenkel [15] and A. Friedman and B. Turkington [29] in the three-dimensional case. For simplicity, we will only refer to the two-dimensional problem.

Proposition 3.1 : Assume f satisfies (f.1)-(f.4) and let $\eta > 0$ be given. Let the function u and the parameters $W > 0$ and $k \geq 0$ satisfy (II), and set

$$(3.9) \qquad F(z) = \int_0^z f(s)ds \quad \forall z \in \mathbb{R} ,$$

$$(3.10) \qquad \mu = \int_\Pi F(u-Wr-k)dx ,$$

$$(3.11) \qquad \chi = \int_\Pi f(u-Wr-k)dx ,$$

$$(3.12) \qquad r_c = \frac{1}{\chi}\int_\Pi rf(u-Wr-k)dx.$$

Then one has :

(3.13) $\gamma \equiv Wr_c\chi - 2\lambda\mu = 0.$

Remark 3.5 : For general identities of this kind, the reader is referred to M.J. Esteban and P.L. Lions [25]. The authors are thankful to J. Norbury for having pointed out this identity and its role in this framework. We only describe here a heuristic proof (for a rigourous derivation, see [25]). ∎

Proof of Proposition 3.1 : Let $\sigma > 0$ be given, and define v_σ as follows :

$$v_\sigma(r,z) = u(r/\sigma, z/\sigma) \quad \forall x = (r,z) \in \Pi.$$

Now, if u is a solution of (II), we know that for the function

$$\Phi(\sigma) = \frac{1}{2}\int_\Pi |\nabla v_\sigma|^2\, dx - \lambda\int_\Pi F(v_\sigma - Wr - k)\,dx$$

one has

$$0 = \Phi'(\sigma)\big|_{\sigma=1} \equiv W\chi\, r_c - 2\lambda\mu.$$

This proves (3.13). ∎

4. A GENERAL MODEL PROBLEM IN A BOUNDED DOMAIN.

In this section and up to section 7, we consider a general class of problems that model problems of the type (I)-W or (I)-k in bounded domains.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\Gamma = \partial\Omega$. Let L be the second order self-adjoint operator defined by

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u$$

where $a_{ij} \in C^{1,\alpha}(\bar{\Omega})$, $c \in C^{0,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$, $c \geq 0$, $a_{ij} = a_{ji}$ and

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \sigma|\xi|^2, \quad \forall x \in \bar{\Omega}, \quad \forall \xi \in \mathbb{R}^N$$

where $\sigma > 0$ is an ellipticity constant. Let $a(\cdot,\cdot)$ be the Dirichlet bilinear form in $H_0^1(\Omega)$ associated with L :

$$a(u,v) = \int_\Omega \Big(\sum_{i,j=1}^{N} a_{ij}(x)\,\frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_j}\Big)dx + \int_\Omega c(x)uv\,dx$$

$$= \int_\Omega Lu\cdot v\,dx\ ,\quad \forall u,v \in H_o^1(\Omega).$$

Let $g : \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ and $\rho : \overline{\Omega}\to\mathbb{R}_+$ be given functions. We consider the following problem

$$(4.1)\qquad \begin{cases} \text{To find } u\in H^2(\Omega) \text{ and } \lambda\in\mathbb{R} \text{ satisfying} \\ Lu = g(x,u-\lambda\rho(x)) \text{ in } \Omega \\ u=0 \text{ on } \Gamma \\ a(u,u) = \eta \end{cases}$$

where $\eta > 0$ is a given number. We assume that g satisfies assumptions analogous to (f.1)-(f.4), namely :

(g.1) $\quad g : \overline{\Omega}\times\mathbb{R}\to[0,+\infty)$ is continuous

(g.2) $\quad g(x,s) = 0\ ,\ \forall s\le 0\ ,\ \forall x\in\overline{\Omega}$

(g.3) $\quad 0<g(x,s)<g(x,s'),\ \forall x\in\Omega,\ \forall 0<s<s',$

$$(g.4)\qquad \begin{cases} \lim_{s\to+\infty} g(x,s)/s^p = 0, \text{ uniformly with respect to} \\ x\in\overline{\Omega}, \text{ with } 1<p<+\infty \text{ if } N\le 2 \text{ and } 1<p<(N+2)/(N-2) \\ \text{if } N\ge 3. \end{cases}$$

$$(g.5)\qquad \begin{cases} \lim_{s\to+\infty} g(x,s) = +\infty \text{ uniformly with respect to } x\in\overline{\Omega}_o \\ \text{for some non empty open subset } \Omega_o\subset\subset\Omega. \end{cases}$$

Lastly, we assume

(4.2) $\quad \rho\in C^o(\overline{\Omega}),\quad \rho>0 \text{ in } \Omega.$

Remark 4.1 : Taking $L=\mathcal{L}$ (resp. $L = -\Delta$), $\bar{\Omega}\subset\Pi$ and $g(x,s) = f(r,s-\frac{W}{2}r^2)$ (resp. $g(x,s) = f(s-Wr)$), where $x = (r,z)$, and $\rho\equiv 1$, one obtains the analog to problem (I)-k (resp. (II)-k) on the bounded domain Ω. Alternatively, taking $g(x,s) = f(r,s-k)$ (resp. $g(x,s) = f(s-k)$) and $\rho(x) = \frac{r^2}{2}$ (resp. $\rho(x)=r$) one obtains the analog of problem (I)-W (resp. (II)-W) in Ω. ∎

Remark 4.2 : Some (hydrodynamically interesting) vorticity functions f for (say) the vortex pair problem which lead (as in Remark 4.1) to a function g satisfying (g.1)-(g.5) are the following :

$$f(s) = \begin{cases} (\frac{s}{1+\beta})^{\beta} \ \underline{\text{for}}\ s>0,\ 0<\beta\le 1, \\ 0\ \underline{\text{for}}\ s\le 0. \end{cases} \tag{4.3}$$

$$f(s) = \begin{cases} 1+\beta(s-\varepsilon)\ \underline{\text{for}}\ s>\varepsilon,\ 0<\beta\le\varepsilon\le 1, \\ s/\varepsilon\ \underline{\text{for}}\ 0<s\le\varepsilon, \\ 0\ \underline{\text{for}}\ s\le 0. \end{cases} \tag{4.4}$$

Clearly, both functions verify assumptions (f.1)-(f.4). ∎

For the problem (4.1) we obtain the following existence result.

Theorem 4 : Assume g satisfies (g.1)-(g.5), then problem (4.1) possesses a solution $u\in H^2(\Omega)$ and $\lambda\in\mathbb{R}$ for any given $\eta>0$.

Remark 4.3 : By a method which is different from the one below and which is less "constructive", using a topological degree argument, the same result is true under more general hypotheses. In particular, (g.3) is not needed (see H. Berestycki [7,8,9]).

Since it will be used in the next Section, we give the outline of the argument which is based on Schauder's fixed point theorem.

Proof of Theorem 4.

Step 1 : Fixed point formulation of the problem. Let $\lambda\in\mathbb{R}$ and $v\in H^1_0(\Omega)$; one defines $u = S(v,\lambda)$ to be the solution of

$$\begin{cases} Lu = g(x,v-\lambda\rho)\ \text{in}\ \Omega, \\ u=0\ \text{on}\ \Gamma. \end{cases} \tag{4.5}$$

Using (g.1)-(g.4), one shows that $S : H_0^1(\Omega) \times \mathbb{R} \to H_0^1(\Omega)$ is a compact operator. Furthermore, if $\lambda \leq \mu$, then $S(v,\mu) \geq S(v,\lambda) \geq 0$ a.e. in Ω, $\forall v \in H_0^1(\Omega)$ (this is a direct consequence of (g.3) and the maximum principle). Let us now define

$$\alpha(v,\lambda) = \int_\Omega g(x,v-\lambda\rho)\, S(v,\lambda)\ dx.$$

$\alpha : H_0^1(\Omega) \times \mathbb{R} \to [0,+\infty)$ is well defined and continuous. It is easily seen that $\lambda \to \alpha(v,\lambda)$ is strictly decreasing provided $\alpha(v,\lambda) > 0$.

We require the following

<u>Lemma 4.1</u> : For any $v \in H_0^1(\Omega)$, one has :

(4.6) $$\lim_{\lambda \searrow -\infty} \nearrow S(v,\lambda) = +\infty\ ,$$

(4.7) $$\lim_{\lambda \nearrow +\infty} \searrow S(v,\lambda) = 0.$$

<u>Proof of Lemma 4.1</u> : We only present the argument assuming $v \in L^\infty(\Omega)$. The general case just requires simple technical modifications and uses the Sobolev embedding Theorems and the Lebesgue convergence Theorems (see [9]).

As $\lambda \to -\infty$, one has $v-\lambda\rho \to +\infty$ everywhere on Ω. By the maximum principle, $S(v,\lambda) \geq \phi$ for some fixed and continuous function $\phi > 0$ in Ω. On the other hand, we know that $g(x,v-\lambda\rho) \to +\infty$ uniformly on $\overline{\Omega}_0$. Hence,

$$\alpha(v,\lambda) \geq \int_{\Omega_0} g(x,v-\lambda\rho)\ S(v,\lambda)\ dx$$

and thus,

(4.8) $$\alpha(v,\lambda) \geq (\min_{\Omega_0} \phi) \int_{\Omega_0} g(x,v-\lambda\rho)\ dx.$$

Since the right hand side in (4.8) converges to $+\infty$ as $\lambda \searrow -\infty$, we derive (4.6).

When $\lambda \to +\infty$, one has $(v-\lambda\rho)^+ \to 0$ a.e. on Ω, where $s^+ = \max(s,0)$. Therefore, $g(x,v-\lambda\rho) \to 0$ in $L^\infty(\Omega)$ and $S(v,\lambda) \to 0$ in $L^\infty(\Omega)$. Hence, clearly, $\alpha(v,\lambda) \searrow 0$ as $\lambda \nearrow +\infty$. ■

Lemma 4.1 and the fact that $\lambda \to \alpha(v,\lambda)$ is strictly decreasing wherever it is positive allow us to define $\lambda = \Lambda(v)$ from the equation

$$\alpha(v,\Lambda(v)) = \eta \quad \forall v \in H_o^1(\Omega).$$

One can easily prove that the mapping $v \to \Lambda(v)$ is continuous : $H_o^1(\Omega) \to \mathbb{R}$. We now set

$$Tv = S(v,\Lambda(v)).$$

Step 2 : Properties of T. Problem (4.1) is equivalent to the fixed point equation

$$u = Tu. \tag{4.9}$$

Indeed, (4.9) reads $u = S(u,\lambda)$ with $\lambda = \Lambda(u)$, whence, by definition,

$$\begin{cases} Lu = g(x,u-\lambda\rho) \text{ in } \Omega, \\ u=0 \text{ on } \Gamma, \\ a(u,u) = \int_\Omega Lu\cdot u\, dx = \int_\Omega g(x,u-\lambda\rho)\ S(u,\lambda) = \alpha(u,\Lambda(u)) = \eta. \end{cases}$$

One can show using a stronger version of Lemma 4.1 that $T : H_o^1(\Omega) \to H_o^1(\Omega)$ is a compact operator. We omit the details of the proof here (see [7] for details and [9] for a related result).

Step 3 : Conclusion. Let $K = \{u \in H_o^1(\Omega)\ ;\ a(u,u) \leq \eta+1\}$. Then, K is a closed convex and bounded set in $H_o^1(\Omega)$. For any $v \in H_o^1(\Omega)$, $u=Tv$ verifies

$$\begin{cases} a(u,u) = \int_\Omega Lu\cdot u\, dx = \int_\Omega g(x,v-\lambda\rho) u\, dx = \\ = \int_\Omega g(x,v-\lambda\rho) S(v,\lambda)\, dx = \alpha(v,\Lambda(v)) = \eta \end{cases}$$

(for $\lambda = \Lambda(v)$). Thus, for any $v \in H_o^1(\Omega)$, one has $Tv \in \{u \in H_o^1(\Omega)\ ;\ a(u,u)=\eta\}$ and therefore, T maps K into itself. T being a compact operator has a fixed point u by Schauder's fixed point Theorem. Hence, u is a solution of (4.1). ∎

Remark 4.4 : From the above proof it is clear that one can consider more general problems of the kind

$$\begin{cases} Lu = \tilde{g}(x,u,\lambda) \text{ in } \Omega, \\ u=0 \text{ on } \Gamma, \\ \int_\Omega Lu\cdot u \, dx = \eta , \end{cases}$$

under suitable assumptions on $\tilde{g} : \overline{\Omega}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}_+$. ∎

5. THE DISCRETE VERSION OF THE MODEL PROBLEM.

In this Section and in the next one, we develop a finite element approach for the solution of problem (4.1). For the sake of simplicity, we shall assume here that $\overline{\Omega}$ is a non-degenerate polyhedron of $\mathbb{R}^N$. Let $\mathcal{H}$ be a generalized sequence in $\mathbb{R}^s\setminus\{0\}$ converging to zero, to which is associated a family of triangulations $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ of Ω. We assume that $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ is a regular family in the sense of P. Ciarlet [20] (see next Section).

For $K\in\mathcal{T}_h$, we denote by $P_\ell(K)$ the space of polynomial functions on K whose degree does not exceed $\ell\in\mathbb{N}^*$. The notation P(K) in the sequel will refer either to $P_1(K)$ or to $P_2(K)$, or else to some subspace of $P_\ell(K)$ which will be specified later on. For any $h\in\mathcal{H}$, we consider the spaces

(5.1) $$W_h = \{w_h\in C^0(\overline{\Omega}) \; ; \; w_{h|K}\in P(K) \; , \; \forall K\in\mathcal{T}_h\},$$

and

(5.2) $$V_h = \{w_h\in W_h \; ; \; w_{h|\Gamma} = 0\}.$$

V_h is a finite dimensional subspace of $H^1_o(\Omega)$. Let $m = m(h) = \dim V_h$. Let us denote by $<\cdot,\cdot>$ the L^2 scalar product - or its restriction to V_h - and by $(\cdot,\cdot)$ the usual scalar product in $\mathbb{R}^m$. Let $\{w_h^1,\dots,w_h^m\}$ be the classical basis of V_h. Recall that the w_h^j are nonnegative functions (see e.g. P. Ciarlet [20]).

It is quite natural to set up a finite dimensional approximation of (4.1) by means of the following discrete problems :

$$(5.3)_h \qquad \begin{cases} \text{Find } u_h \in V_h \text{ and } \lambda_h \in \mathbb{R} \text{ satisfying} \\ a(u_h,v_h) = \langle g(\cdot,u_h-\lambda_h\rho),v_h\rangle \ , \\ \forall v_h \in V_h, \text{ and } a(u_h,u_h) = \eta \ . \end{cases}$$

Consider the $m\times m$ matrix

$$(5.4) \qquad A_h = (a(w^i,w^j))_{i,j=1}^{m} \ .$$

A_h is always positive definite, symmetric. We further assume that the approximation is constructed in such a way as to ensure that the A_h are of "monotone class" (or "monotone"), that is, satisfy

$$(5.5) \qquad \text{All the elements in } A_h^{-1} \text{ are nonnegative.}$$

For the cases we are interested in, namely, $N=2$, $L=-\Delta$ or $L=\mathcal{L}$, the choices of V_h are detailed below in Section 8 and will be seen to yield property (5.5).

For $\zeta_h \in \mathbb{R}^m$ and $\lambda_h \in \mathbb{R}$ denote by $b_h(\zeta_h,\lambda_h)$ the vector in $\mathbb{R}^m$ having components b_h^j defined by :

$$(5.6) \qquad b_h^j(\zeta_h,\lambda_h) = \langle g(\cdot,u_h-\lambda_h\rho),w_h^j\rangle$$

where $u_h = \sum_{j=1}^{m} \zeta_h^j w_h^j$. Then, $(5.3)_h$ is equivalent to the problem

$$(5.7)_h \qquad \begin{cases} \text{To find } \zeta_h \in \mathbb{R}^m \text{ and } \lambda_h \in \mathbb{R} \text{ satisfying} \\ A_h\zeta_h = b_h(\zeta_h,\lambda_h), \\ \langle A_h\zeta_h,\zeta_h\rangle = \eta \ . \end{cases}$$

Using the same type of arguments as the one outlined in Section 3 above for the continuous case, one obtains the following existence result for $(5.7)_h$.

<u>Theorem 5</u> : Assume g satisfies (g.1)-(g.5) and that A_h is of monotone class (5.5). Then there exists a solution (ζ_h,λ_h) of problem $(5.7)_h$.

Proof of Theorem 5 : Form now on, since $h \in \mathcal{H}$ is fixed, we drop everywhere in this section the subscript h. We require a sequence of Lemma that parallels the steps of the proof of Theorem 4 in Section 4.

Lemma 5.1 : For all $\xi \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$, let $\zeta = S(\xi,\lambda)$ denote the unique solution of

$$A\zeta = b(\xi,\lambda) \qquad (\zeta \in \mathbb{R}^m)$$

where A ($=A_h$) and b = (b_h) are defined in (5.4) and (5.6) respectively. Then $(\xi,\lambda) \to S(\xi,\lambda)$ is a continuous map : $\mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$. Furthermore, if $\lambda \leq \mu$, then $S(\xi,\lambda) \geq S(\xi,\mu) \geq 0$; if $\xi^j \geq \hat{\xi}^j$ $\forall j$, then $S(\xi,\lambda) \geq S(\hat{\xi},\lambda)$.

Proof of Lemma 5.1 : A_h is invertible so $\zeta = S(\xi,\lambda)$ is well defined. Since b is continuous it is straightforward that $S(\xi,\lambda) = A^{-1}b(\xi,\lambda)$ is continuous : $\mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$. Now observe that since A verifies (5.5), A^{-1} preserves the natural ordering in $\mathbb{R}^m$. If $\lambda \leq \mu$, then, because the w_j are nonnegative and g satisfies (g.2)-(g.3), it is obvious that $b(\xi,\lambda) \geq b(\xi,\lambda) \geq 0$ and we conclude using property (5.5). Similarly, if $\xi^j \geq \hat{\xi}^j$ $\forall j$; one has $b(\xi,\lambda) \geq b(\hat{\xi},\lambda)$, whence $S(\xi,\lambda) \geq S(\hat{\xi},\lambda)$. ■

Lemma 5.2 : Consider the function $\alpha : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ defined by

$$\alpha(\xi,\lambda) = (b(\xi,\lambda),S(\xi,\lambda))$$

Then, $\alpha \geq 0$, α is continuous, and for any fixed $\xi \in \mathbb{R}^m$, $\lambda \to \alpha(\xi,\lambda)$ is non-increasing and is actually decreasing as soon as $\alpha(\xi,\lambda) > 0$. Moreover,

$$(5.8) \qquad \lim_{\lambda \searrow -\infty} \nearrow \alpha(\xi,\lambda) = +\infty ,$$

$$(5.9) \qquad \lim_{\lambda \nearrow +\infty} \searrow \alpha(\xi,\lambda) = 0.$$

Proof of Lemma 5.2 : Let us first show that $\alpha(\xi,\lambda)$ is decreasing with respect to λ if $\alpha(\xi,\lambda) > 0$. Suppose that $\lambda > \mu$ and $\alpha(\xi,\mu) > 0$. Hence, there exists some j such that $b^j(\xi,\lambda) > 0$ and $S^j(\xi,\mu) > 0$ (where $b=(b^1,\ldots b^m)$ and $S = (S^1,\ldots,S^m)$). Letting $u = \sum_{j=1}^m \xi^j w^j$, it is clear that

$$b^j(\xi,\mu) = \langle g(\cdot,u-\mu\rho),w_j\rangle > \langle g(\cdot,u-\lambda\rho),w_j\rangle = b^j(\xi,\lambda).$$

Thus, $b^j(\xi,\mu) > b^j(\xi,\lambda)$. Therefore, using $S^j(\xi,\mu) \geq S^j(\xi,\lambda)$ and $S^j(\xi,\mu) > 0$, we derive

$$S^j(\xi,\mu)b^j(\xi,\mu) > S^j(\xi,\lambda)b^j(\xi,\lambda).$$

Since for any $k=1,\ldots,m$, one obviously has

$$S^k(\xi,\mu)\; b^k(\xi,\mu) \geq S^k(\xi,\lambda)\; b^k(\xi,\lambda),$$

we obtain :

$$\alpha(\xi,\mu) > \alpha(\xi,\lambda).$$

For the proof of (5.9) it suffices to observe that $g(\cdot,u-\lambda\rho) \to 0$ uniformly as $\lambda \nearrow +\infty$, whence $b(\xi,\lambda) \to 0$ and $S(\xi,\lambda) \to 0$. We now turn to the proof of (5.8). As $\lambda \searrow -\infty$, then $g(\cdot,u-\lambda\rho) \nearrow +\infty$ uniformly on Ω_o, while $g(\cdot, u-\lambda\rho) \geq \phi$ in Ω, for some fixed function $\phi > 0$ in Ω. One therefore has :

$$(5.10) \qquad b^j(\xi,\lambda) \geq \beta^j > 0,\; j=1,\ldots,m,$$

where $\beta^j = \langle\phi\;,w^j\rangle > 0$. There exists some j_o, $1 \leq j_o \leq m$, such that $\mathrm{supp}\;(w^{j_o}) \cap \Omega_o \neq \emptyset$. Hence,

$$(5.11) \qquad b^{j_o}(\xi,\lambda) = \langle w^{j_o},g(\cdot,u-\lambda\rho)\rangle \to +\infty \text{ as } \lambda \searrow -\infty\;.$$

(Here, $u = \sum_{j=1}^{m} \xi^j w^j$). Since

$$\|S(\xi,\lambda)\| = \|A^{-1}b(\xi,\lambda)\| \geq \|A\|^{-1}\;\|b(\xi,\lambda)\|\;,$$

(5.10) and (5.11) imply $\|S(\xi,\lambda)\| \to +\infty$. But $S^j(\xi,\lambda) \geq 0\;\; \forall j$, whence, using (5.10), one derives :

$$(5.12) \qquad \alpha(\xi,\lambda) \geq \sum_{j=1}^{m} \beta^j S^j(\xi,\lambda) \to +\infty \text{ as } \lambda \searrow -\infty.$$

That is, $\lim_{\lambda \searrow -\infty} \alpha(\xi,\lambda) = +\infty$. ∎

An inspection of the preceding argument shows at once that one has the following refinement of (5.8) and (5.9) :

<u>Lemma 5.3</u> : The limits (5.8) and (5.9) hold uniformly with respect to ξ in any bounded set of $\mathbb{R}^m$. ∎

We are now ready to conclude the proof of Theorem 5. By Lemma 5.2, we can construct a mapping $\Lambda : \mathbb{R}^m \to \mathbb{R}$ by defining $\lambda = \Lambda(\xi)$ to be the unique solution of

$$\alpha(\xi,\Lambda(\xi)) = \eta \; . \tag{5.13}$$

With the aid of Lemma 5.3, it is easily verified that $\Lambda : \mathbb{R}^m \to \mathbb{R}$ is continuous. Let us now define an operator $T : \mathbb{R}^m \to \mathbb{R}^m$ by setting

$$T\xi = S(\xi,\Lambda(\xi)). \tag{5.14}$$

T is continuous and verifies

$$(A(T\xi),T\xi) = \alpha(\xi,\Lambda(\xi)) = \eta \; . \tag{5.15}$$

Let $K = \{\xi \in \mathbb{R}^m ; (A\xi,\xi) \le \eta+1\}$. K is a compact and convex set ; T maps K into itself. Therefore, by Brouwer's fixed point theorem, there exists $\zeta \in \mathbb{R}^m$ verifying

$$\zeta = T\zeta \; . \tag{5.16}$$

Let $\lambda = \Lambda(\xi)$. Then, $\zeta = S(\zeta,\lambda)$ which means

$$A\zeta = b(\zeta,\lambda)$$

and

$$(A\zeta,\zeta) = (AS(\zeta,\lambda),S(\zeta,\lambda)) = \alpha(\zeta,\lambda) = \eta,$$

since $\lambda = \Lambda(\zeta)$. Thus, (ζ,λ) is a solution of $(5.7)_h$ and the proof of Theorem 5 is complete. ■

Remark 5.1 : In very much the same way as above, we can deal with a finite element approach which involves numerical integration ; one then obtains an analogous existence theorem. Actually, the convergence results of next Section are also valid when quadrature schemes are introduced. For details in this direction and related results see E. Fernandez Cara [26]. ■

6. CONVERGENCE OF THE FINITE ELEMENT APPROXIMATION.

The purpose of this section is to prove that the finite element approximation defined by problem $(5.3)_h$ converges to solutions of problem (4.1). Hereafter, we shall assume that the family of triangulations $\{\mathcal{C}_h\}_{h\in\mathcal{H}}$ and spaces $\{V_h\}_{h\in\mathcal{H}}$ satisfy the following usual requirements :

(6.1) $$\{\mathcal{C}_h\}_{h\in\mathcal{H}} \text{ is a regular family of triangulations of } \Omega.$$

The notion of "regular family" is defined in P. Ciarlet [20]. It means that the family has the following two properties. Let us denote

$$\delta(K) = \text{diameter of } K$$
$$\rho(K) = \text{diameter of the largest ball contained in } K.$$

Then,

(6.2) $$\lim_{\substack{h\to 0\\ h\in\mathcal{H}}} \{\max_{K\in\mathcal{C}_h} \delta(K)\} = 0$$

and

(6.3) $$\exists\, C>0 \text{ such that } \delta(K)/\rho(K) \le C,\ \forall K\in\mathcal{C}_h,\ \forall h\in\mathcal{H}.$$

Let $q = \frac{2N}{N+2}$ if $N \ge 3$ and $1 < q < \infty$ if $N \le 2$. Let $f \in L^q(\Omega)$ and let $w = Qf \in W^{2,q}(\Omega)$ denote the solution of the Dirichlet problem

(6.4) $$\begin{cases} Lw = f \text{ in } \Omega, \\ w = 0 \text{ on } \Gamma. \end{cases}$$

Let $w_h = Q_h f \in V_h$ denote the solution of the problem

(6.5) $$\begin{cases} a(w_h,v_h) = \langle f,v_h\rangle \quad \forall v_h \in V_h \\ w_h \in V_h \end{cases}$$

Then, we assume that the following property holds :

(6.6) $$\forall f \in L^q(\Omega),\quad \lim_{\substack{h\to 0\\ h\in\mathcal{H}}} \|w-w_h\|_{H^1_o(\Omega)} = 0$$

That is, we assume that the finite element approximation is set up in such a way that the approximations of the linear Dirichlet problem converge. This property is classical in most of the practical situations considered in the next sections. For the three dimensional problem, however, we will check (6.6) in Section 8 for a particular choice of P_K.

The main result of this section is the following

Theorem 6 : Assume that g satisfies (g.1)-(g.4), that the family of triangulations $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ is regular, ((6.1)), and that the family of spaces $\{V_h\}_{h\in\mathcal{H}}$ verifies (5.5) and (6.6). Let $u_h \in V_h$ and $\lambda_h \in \mathbb{R}$ be a solution of $(5.3)_h$ (obtained from $(5.7)_h$). There exists a subsequence $\mathcal{H}' \subset \mathcal{H}$ such that $u_{h'} \to u^*$ strongly in $H_o^1(\Omega)$ and $\lambda_{h'} \to \lambda^*$ along $h' \to 0$, $h' \in \mathcal{H}'$. Furthermore, (u^*,λ^*) is a solution of (4.1).

Proof of Theorem 6 : Since $a(u_h,u_h) = \eta$, the sequence $\{u_h\}_{h\in\mathcal{H}}$ is bounded in $H_o^1(\Omega)$. Hence, there exists a subsequence denoted again for simplicity by $\{u_h\}_{h\in\mathcal{H}}$ such that

(6.7) $\qquad u_h \to u$ weakly in $H_o^1(\Omega)$ as $h \to 0$.

Since $pq < 2^* = \frac{2N}{N-2}$ if $N \geq 3$, it follows from (6.7) by the Sobolev embedding theorem that (for any $N \geq 1$) one has

(6.8) $\qquad u_h \to u^*$ strongly in $L^{pq}(\Omega)$ as $h \to 0$.

Let S,Λ and α be as in section 4, while S_h,Λ_h,α_h are the corresponding finite dimensional mappings introduced in Section 5 (compare Lemmas 5.1 and 5.2).

The proof of Theorem 6 will be divided into the next three Lemmas. The first one is a classical consequence of (6.6), where we use the notations introduced for (6.4)-(6.6).

Lemma 6.1 : Let $\{f_h\}$ be a sequence in $L^q(\Omega)$ such that $f_h \to f$ strongly in $L^q(\Omega)$. From (6.6) it follows that $Q_h f_h \to Qf$ strongly in $H_o^1(\Omega)$.

Proof of Lemma 6.1 : Let $w_h = Q_h f$ and $\hat{w}_h = Q_h f_h$. From (6.6) one knows that $w_h \to Qf$ strongly in $H_o^1(\Omega)$; w_h and $\hat{w}_h$ are defined by

$$\left.\begin{aligned} a(w_h,v_h) &= \langle f,v_h\rangle \\ a(\hat{w}_h,v_h) &= \langle f_h,v_h\rangle \end{aligned}\right\} \forall v_h \in V_h$$

and $w_h,\hat{w}_h \in V_h$. Hence, one derives from the preceding variational equalities :

$$a(\hat{w}_h-w_h,\hat{w}_h-w_h) = \langle f-f_h,\hat{w}_h-w_h\rangle \ .$$

Consequently, by Hölder's inequality, one has

$$\text{(6.9)} \qquad \|\hat{w}_h-w_h\|^2_{H^1_o} \le C\,\|f-f_h\|_{L^q}\ \|\hat{w}_h-w_h\|_{L^r} \quad ,$$

where $C>0$ is a constant (C^{-1} is the ellipticity constant) and where $r=q'$, $\frac{1}{q}+\frac{1}{r}=1$. When $N\ge 3$, one has $r=\frac{2N}{N-2} = 2$, whence, in all cases, by the Sobolev imbedding theorem, $H^1_o(\Omega) \subset L^r(\Omega)$ and one obtains from (6.9) that

$$\text{(6.10)} \qquad \|\hat{w}_h-w_h\|_{H^1_o} \le C\,\|f_h-f\|_{L^q} ,$$

for some other constant $C>0$. Therefore, since $w_h \to Qf$ in $H^1_o(\Omega)$, (6.10) implies $\hat{w}_h = Q_hf_h \to Qf$ strongly in $H^1_o(\Omega)$. ∎

<u>Lemma 6.2</u> : Let $\{v_h\} \subset H^1_o(\Omega)$ and $\{\lambda_h\} \subset \mathbb{R}$ be sequences such that $v_h \to v$ weakly in $H^1_o(\Omega)$ and $\lambda_h \to \lambda$. Then, $S_h(v_h,\lambda_h) \to S(v,\lambda)$ strongly in $H^1_o(\Omega)$.

<u>Proof of Lemma 6.2</u> : Let us first recall that S_h was defined as an operator : $\mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ $(m=m(h))$ in Section 5. Here, however, we will consider S_h as an operator $V_h \times \mathbb{R} \to V_h$, by identifying V_h with $\mathbb{R}^m$ through the isomorphism

$$v_h \in V_h \leftrightarrow \zeta \in \mathbb{R}^m \text{ with } v_h = \sum_{j=1}^{m} \zeta^j w_h^j .$$

Now $w_h = S_h(v_h,\lambda_h)$ is defined by :

$$\text{(6.11)} \qquad \left\{\begin{aligned} & a(w_h,z_h) = \langle g(\cdot,v_h-\lambda_h\rho),z_h\rangle \ , \ \forall z_h \in V_h \ , \\ & w_h \in V_h \end{aligned}\right.$$

Let $g_h = g(\cdot, v_h - \lambda_h \rho)$. Then, with the notations of this Section, (6.11) reads $w_h = Q_h g_h$. Now, as was already observed, from $v_h \to v$ in $H_o^1(\Omega)$, one has $v_h \to v$ in $L^{pq}(\Omega)$ and therefore

$$(6.12) \qquad v_h - \lambda_h \rho \to v - \lambda \rho \text{ strongly in } L^{pq}(\Omega).$$

Hypothesis (g.4) implies

$$(6.13) \qquad 0 \leq g(x,s) \leq C + C|s|^p, \ \forall x \in \overline{\Omega}, \ \forall s \in \mathbb{R}.$$

($C > 0$ will continue to denote generically in the sequel positive constants). It is classical that (6.13) implies that the Nemytskii's operator $w \to g(\cdot, w)$ acts continuously from $L^{pq}(\Omega)$ into $L^q(\Omega)$. Hence, by (6.12) and (6.13), one obtains

$$(6.14) \qquad g_h \to g_o = g(\cdot, v - \lambda \rho) \text{ strongly in } L^q(\Omega).$$

Thus, by Lemma 6.1 one gets from (6.14) that

$$(6.15) \qquad w_h = Q_h g_h \to w = Q g_o = S(v, \lambda) \text{ in } H_o^1(\Omega).$$

The proof of Lemma 6.2 is thereby complete. ■

<u>Lemma 6.3</u> : Let $v_h \in V_h$, $\forall h \in \mathcal{H}$. Suppose that $\{v_h\}$ remains bounded in $H_o^1(\Omega)$ as $h \to 0$. Then $\Lambda_h(v_h)$ is bounded in $\mathbb{R}$.

<u>Proof of Lemma 6.3</u> : The proof is by contradiction. Let us assume that either

$$(6.16) \qquad \lambda_h = \Lambda_h(v_h) \to +\infty \text{ as } h \to 0,$$

or

$$(6.17) \qquad \lambda_h = \Lambda_h(v_h) \to -\infty \text{ as } h \to 0.$$

After extraction of a subsequence of $\{v_h\}_{h \in \mathcal{H}}$, which is again denoted by $\{v_h\}_{h \in \mathcal{H}}$, we can assume that

$$(6.18) \qquad \begin{cases} v_h \to v \text{ weakly in } H_o^1(\Omega) \\ \text{and} \\ v_h \to v \text{ strongly in } L^{pq}(\Omega). \end{cases}$$

We first consider assumption (6.16). Since $v_h \to v$ in L^{pq}, we know by a converse to Legesgue's dominated convergence theorem (see e.g. N. Bourbaki [14]) that there is a subsequence, denoted once more by $\{v_h\}$ and a $\phi \in L^{pq}(\Omega)$, $\phi \geq 0$, such that

$$(6.19) \qquad 0 \leq |v_h| \leq \phi \text{ a.e. in } \Omega$$

Hence, using (g.2)-(g.3) and (6.13) one derives from (6.19), as soon as $\lambda_h \geq 0$, that

$$(6.20) \qquad 0 \leq g(x, v_h - \lambda_h \rho) \leq g(x, \phi) \leq C + C|\phi|^p.$$

The right hand side in (6.20) is a fixed function in $L^q(\Omega)$. Therefore, since $v_h - \lambda_h \rho \to -\infty$ a.e. in Ω as $h \to 0$, one has by Legesgue's theorem :

$$(6.21) \qquad g(\cdot, v_h - \lambda_h \rho) \to 0 \text{ strongly in } L^q(\Omega).$$

Now, by Lemma 5.1 one knows that as soon as $\lambda_h \geq 0$:

$$(6.22) \qquad 0 \leq S_h(v_h, \lambda_h) \leq S_h(v_h, 0).$$

Lemma 6.2 shows that $S_h(v_h, 0) \to S(v, 0)$ strongly in $H_o^1(\Omega)$, whence strongly in $L^{q'}(\Omega)$ (recall that $H_o^1(\Omega) \subset L^{q'}(\Omega)$). Therefore, it follows from (6.22) that

$$(6.23) \qquad S_h(v_h, \lambda_h) \text{ is bounded in } L^{q'}(\Omega)$$

Combining (6.21) and (6.23) now yields :

$$(6.24) \qquad \alpha_h(v_h, \lambda_h) = \int_\Omega S_h(v_h, \lambda_h) g(\cdot, v_h - \lambda_h \rho) dx \to 0 \text{ as } h \to 0.$$

Since $\alpha_h(v_h, \lambda_h) = \eta$ (for $\lambda_h = \Lambda_h(v_h)$), (6.24) is absurd and (6.16) is ruled out.

Let us now assume that (6.17) holds. As before, by (6.19), we know that $v_h \geq -\phi$ a.e. in Ω. It is straightforward to observe that S_h can actually be defined as an operator $S_h : H_o^1(\Omega) \times \mathbb{R} \to V_h$ which extends the operator $S_h : V_h \times \mathbb{R} \to V_h$ previously considered. Indeed, for $(u,\lambda) \in H_o^1(\Omega) \times \mathbb{R}$, define the vector $b(u,\lambda) \in \mathbb{R}^m$ by setting $b^j(u,\lambda) = \langle g(\cdot,u-\lambda\rho),w_h^j\rangle$ as in (5.6). We then set $S_h(u,\lambda) = A_h^{-1} b(u,\lambda) \in V_h$. As in Lemma 5.1, one easily checks that if $u \leq \hat{u}$, and $\lambda \leq \mu$, then $S_h(\hat{u},\lambda) \geq S_h(u,\mu)$. (This is a direct consequence of (g.3), of the "monotone" character of A_h and of the fact that $w_h^j \geq 0$).

Now, since $v_h \geq -\phi$, one has $S_h(v_h,\lambda_h) \geq S_h(-\phi,\lambda_h)$. Choosing h sufficiently small, one also has $\lambda_h \leq \Lambda(-\phi)$ and therefore,

(6.25) $$S_h(v_h,\lambda_h) \geq S_h(-\phi,\Lambda(-\phi)).$$

Now, observe that $S_h(-\phi,\Lambda(-\phi)) = Q_h(g(\cdot,-\phi+\Lambda(-\phi)\rho))$. Hence, by (6.6), $S_h(-\phi,\Lambda(-\phi)) \to S(-\phi,\Lambda(-\phi)) \geq 0$, as $h \to 0$. By the maximum principle, since $S(-\phi,\Lambda(-\phi)) \not\equiv 0$ (for $\alpha(-\phi,\Lambda(-\phi)) = \eta$), one actually has $S(-\phi,\Lambda(-\phi)) > 0$ a.e. in Ω. We also know that

(6.26) $$g(\cdot,v_h-\lambda_h\rho) \geq g(\cdot,-\phi-\lambda_h\rho) \text{ a.e. in } \Omega.$$

Since $\lambda_h \to -\infty$ as $h \to 0$, it follows that

(6.27) $$g(\cdot,v_h-\lambda_h\rho) \to +\infty \text{ a.e. in } \Omega_o.$$

But, since $\lambda_h = \Lambda_h(v_h)$

(6.28) $$\eta = \alpha_h(v_h,\lambda_h) \geq \int_{\Omega_o} g(\cdot,v_h-\lambda_h\rho)S_h(v_h,\lambda_h)dx.$$

By (6.25) and (6.27), $g(\cdot,v_h-\lambda_h\rho)S_h(v_h,\lambda_h) \to +\infty$ a.e. in Ω_o, and (6.28) is impossible. Hence, (6.17) is absurd and the proof of Lemma 6.3 is complete. ∎

<u>Conclusion of the proof of Theorem 6</u>. By Lemma 6.3 we know that $\lambda_h = \Lambda_h(u_h)$ remains bounded in $\mathbb{R}$. Thus, we may assume that $\lambda_h \to \lambda^* \in \mathbb{R}$ as $h \to 0$. Using (6.7) and (6.8), we obtain from Lemma 6.2 that

$$S_h(u_h,\lambda_h) \to S(u^*,\lambda^*) \text{ strongly in } H_o^1(\Omega).$$

Since $u_h = S_h(u_h,\lambda_h)$, we see that $u_h \to u^*$ strongly in $H_o^1(\Omega)$ and furthermore,

$$u^* = S(u^*,\lambda^*).$$

One then also has $a(u_h,u_h) \to a(u^*,u^*)$. But

$$a(u_h,u_h) = \langle g(\cdot,u_h-\lambda_h\rho),u_h\rangle$$

$$= \langle g(\cdot,u_h-\lambda_h\rho),S_h(u_h,\lambda_h)\rangle = \alpha_h(u_h,\lambda_h) = \eta$$

for $\lambda_h = \Lambda_h(u_h)$. Hence, (u^*,λ^*) verify

$$Lu^* = g(\cdot,u^*-\lambda^*\rho) \text{ in } \Omega,$$

$$u^* = 0 \text{ on } \Gamma,$$

$$a(u^*,u^*) = \eta .$$

That is, (u^*,λ^*) is a solution of problem 4.1. The proof of Theorem 6 is thereby complete. ∎

7. SOME ALGORITHMS.

In this Section, we list several iterative schemes for the resolution of (4.1). Although these algorithms are used to solve the discrete problems $(5.3)_h$, for the sake of simplicity, we only formulate the methods directly for the continuous case, i.e. (4.1). It is straightforward to adapt the following constructions for the discrete problems.

7.1. Fixed point type algorithms.

In view of the fixed point formulation $u = Tu$ of Problem 4.1 introduced in Section 4, it is quite natural to consider the following iterative method (we use the notations of Section 4) :

Algorithm (A.1)

(a) Let $u^o \in H_o^1(\Omega)$ be an arbitrary function.

(b) For $n \in N$ and $u^n \in H_o^1(\Omega)$ define $\lambda^{n+1} = \Lambda(u^n)$ and $u^{n+1} = S(u^n,\lambda^{n+1})$ that is, $u^{n+1} = T(u^n)$.

A slightly modified procedure, more explicit, reads :

Algorithm (A.2)

(a) Take $v^o \in H^1_o(\Omega)$ and $\mu^o \in \mathbb{R}$.

(b) For $n \geq 0$, $v^n \in H^1_o(\Omega)$ and $\mu^n \in \mathbb{R}$, define $v^{n+1} = S(v^n,\mu^n)$ and μ^{n+1} as the unique real number such that

$$\int_\Omega v^{n+1} \cdot g(x, v^{n+1} - \mu^{n+1}\rho)dx = \eta \ .$$

Clearly, the sequence $\{u^n\}$ defined by (A.1) is bounded in $H^1_o(\Omega)$ and therefore (compare with the proof of Lemma 6.3), $\{\lambda^n\}$ is also bounded. The question of local or global convergence results for the algorithms (A.1) or (A.2) are nevertheless entirely open. If $\{u^{n_j}\}$ is a subsequence of $\{u^n\}$ which converges weakly to u in $H^1_o(\Omega)$ and $\lambda^{n_j} \to \lambda$, then, there exists a subsequence of u^{n_j+1} which converges to $u^* \in H^1_o(\Omega)$. The difficulty then rests in showing that $u = u^*$. We want to emphasize that in the practical computations, both (A.1) and (A.2) are rapidly convergent, and this independently from the choice of the starting function u^o or v^o.

Remark 7.1 : For a problem arising in plasma physics, with a formulation related to (4.1), but with another normalization condition, a global convergence result has been established by H. Berestycki & H. Brézis [11] for an iterative scheme analogous to (A.1). For computations concerning this problem, the reader is referred to the work of M. Sermange [34]. Concerning the nonlinear eigenvalue problem (I)-λ or (II)-λ, one can also obtain global convergence results for certain algorithms (see M. Berestycki & C. Stuart [13,14]). The plasma physics problem in [11] and problems (I)-λ, (II)-λ share the feature of having a variational structure. This allows one to use the "energy" functional as a "Ljapunov function" for the sequence (see [11,13,14] for the details). In the problems we consider here, however, there is no apparent variational structure and the preceding methods fail to apply. ■

Remark 7.2 : In practical situations, it has been observed that the rate of convergence of the algorithms (A.1) and (A.2) can be improved when relaxation parameters are introduced. ■

7.2. Kitchen type algorithms. A somewhat related iterative scheme is the following (see Kitchen [32] and M. Sermange [48]).

Algorithm (A.3)

(a) Take $u^o \in H^1_o(\Omega)$.

(b) For $n \in \mathbb{N}$ and $u^n \in H^1_o(\Omega)$ define

(b_1) $u^{n+1/3} = T(u^n)$

(b_2) $u^{n+2/3} = T(u^{n+1/3})$

(b_3) $u^{n+1} = (1-\alpha)u^n + 2\alpha u^{n+1/3} - \alpha u^{n+2/3}$

where $\alpha > 0$ is a (small enough) fixed positive constant.

Here again, the question of local or global convergence results is essentially open. If u is an isolated solution of (4.1), and under some particular assumptions on the spectrum of T'(u) (Fréchet derivative of T at u) one can adapt here a method of Sermange to show a local convergence result (see E. Fernandez Cara [27] for the details).

7.3. Least squares formulation. Define

$$J(v) = \frac{1}{2}\, a(v-Tv, v-Tv)$$

It is obvious that (4.1) is equivalent to the variational problem

$$\text{(A.4)} \qquad \text{Minimize } \{J(v) \ ; \ v \in H^1_o(\Omega)\},$$

for $J(u) = 0$ when u is the solution of (4.1).

This point of view on (4.1) allows one to use the variational techniques of "gradient type" for this problem. One can thus write down some iterative schemes in this spirit. We omit the details here ; a detailed description of this method and proofs are given in E. Fernandez Cara [26] (see also R. Glowinski [31, Chapter 7] for a discussion of least square methods for solving nonlinear boundary value problems).

7.4. Ordering methods. Let E denote the product space $E = H^1_o(\Omega) \times \mathbb{R}$. An ordering is defined on E as follows : For $e_1 = (v_1, \mu_1)$, $e_2 = (v_2, \mu_2)$ one sets $e_1 \leq e_2$ if and only if $v_1 \leq v_2$ a.e. in Ω and $\mu_1 \geq \mu_2$. E is then seen to be an ordered Banach space. For $e = (v, \mu)$ define an operator $N : E \to E$ by setting

$$N(e) = N(v,\mu) = (S(v,\mu), \Lambda(v)).$$

Then, $N = N_1+N_2$ with $N_1(v,\mu) = (S(v,\mu),0)$ and $N_2(v,\mu) = (0,\Lambda(v))$. From the results of Section 4 above, it is easily seen that N_1 and N_2 are compact operators. Furthermore, N_1 is isotone, and N_2 is antitone, that is $e_1 < e_2 \Longrightarrow N_1(e_1) < N_1(e_2)$ while $e_1 < e_2 \Longrightarrow N_2(e_1) > N_2(e_2)$.

Now suppose that there exists a pair $e_o, f_o \in E$ with the property that

$$(7.1) \qquad e_o \leq N_1(e_o)+N_2(f_o) \leq N_1(f_o)+N_2(e_o) \leq f_o.$$

Define two sequences $\{e_n\}$ and $\{f_n\}$ in E by setting

$$(7.2) \qquad \begin{cases} e_{n+1} = N_1(e_n)+N_2(f_n), \\ f_{n+1} = N_1(f_n)+N_2(e_n). \end{cases}$$

One can then prove using (7.1) (see e.g. L. Collatz [22]) that N maps the order interval $[e_n,f_n] = \{f \in E \; ; \; e_n < f < f_n\}$ into itself and that there exists a solution $e = (u,\lambda)$ of (4.1), that is $e = N(e)$, such that $e_n < e < f_n$, $\forall n$.

These considerations lead one to the following algorithm :

<u>Algorithm (A.5)</u>

(a) Let $u^o, v^o \in H_o^1(\Omega)$ and $\lambda^o, \mu^o \in \mathbb{R}$ be such that

$$(7.3) \qquad \begin{cases} u^o \leq S(u^o,\lambda^o) \leq S(v^o,\mu^o) \leq v^o \text{ a.e. } \Omega \\ \text{and} \\ \lambda^o \geq \Lambda(v^o) \geq \Lambda(u^o) \geq \mu^o. \end{cases}$$

(b) For given $n \geq 0$, $u^n, v^n \in H_o^1(\Omega)$ and $\lambda^n, \mu^n \in \mathbb{R}$, define

$$\begin{cases} u^{n+1} = S(u^n,\lambda^n) \; , \; \lambda^{n+1} = \Lambda(v^n) \\ v^{n+1} = S(v^n,\mu^n) \; , \; \mu^{n+1} = \Lambda(u^n). \end{cases}$$

Using the results of Section 4 and monotonicity arguments, one can show that

$$(u^n,\lambda^n) \to (u^*,\Lambda(v^*)),$$

$$(v^n,\mu^n) \to (v^*,\Lambda(u^*)).$$

Then, $u^n \leq u^* \leq v^* \leq v^n$ a.e. in Ω and $\lambda^n \geq \Lambda(v^*) \geq \Lambda(u^*) \geq \mu^n$, $\forall n \in \mathbb{N}$ and $u^* = S(u^*,\Lambda(v\ ))$ while $v^* = S(v^*,\Lambda(u^*))$. Lastly, $a(u^*,u^*) \leq \eta \leq a(v^*,v^*)$.

Here again, local convergence results are open. The theoretical difficulty here rests in finding (near an isolated solution) initial data satisfying (7.2) and such that the resulting limits verify $u^* = v^*$ (u^* is then a solution of (4.1).

7.5. Comparison of the algorithms. From a practical numerical viewpoint, the algorithm A.1 has proved to be highly efficient. All the numerical results described in Section 10 have been obtained using this iterative scheme. As far as the results are concerned, all the other methods lead to quite similar results (the output being appreciably the same for all five methods). However, algorithms (A.2)-(A.5) are less efficient in that they require more storage and, generally speaking, they are much slower than (A.1). With respect to the other algorithms, though, the method (A.5) has an interesting property. At each step it provides an upper and lower bound for the solution of (4.1). This can prove to be useful if one wants to estimate with precision the vorticity region in problems (I) and (II).

8. APPLICATIONS TO VORTEX RINGS AND PAIRS PROBLEMS.

It is natural to first approximate Π by a family of bounded domains Π_a which "converge" to Π as $a \to +\infty$. Let us consider the two-dimensional problem (II). As an approximation, choose a positive real parameter a (large enough) and consider the rectangle

$$\Pi_a = \{x \in \Pi;\ x = (r,z),\ r < a,\ |z| < a\}.$$

We now look for a function u_a and a parameter $W_a > 0$ (resp. $k_a \geq 0$) satisfying :

Problem (II_a) (two-dimensional) :

$$(II_a) \qquad \begin{cases} -\Delta u = f(u-Wr-k) \text{ in } \Pi_a, \\ u \in H_o^1(\Pi_a) \ , \ u \in C^1(\Pi_a) \ , \\ \int_{\Pi_a} |\nabla u|^2 \, dx = \eta, \end{cases}$$

where $\eta > 0$ and $k \geq 0$ (resp. $W > 0$) are given. We are thus led to a particular case of (4.1), for which the corresponding discretized problems can be formulated. If we deal with a regular family of triangulation $\mathcal{T}_h$, and set $P_K = P_\ell(K)$ for all $K \in \mathcal{T}_h$ and $\ell=1$ or 2 (say), then condition (6.6) is fulfilled (see e.g. [20]), and we have the strong convergence in $H_o^1(\Pi_a) \times \mathbb{R}$ of at least a subsequence of discrete solutions to the solution of (Π_a), provided all matrices A_h are of monotone class.

Remark 8.1 : Another possibility consists in considering the sets

$$C_a = \{x \in \Pi; |x| < a\} \ ,$$

and the corresponding finite element schemes $(5.3)_h$. ∎

Next we turn to the three-dimensional problem (I). Notice that all the results in Sections 4 to 6 hold as well, if we replace the space $H_o^1(\Omega)$ by $H_o(\Omega)$ (introduced in L.E. Fraenkel and M.S. Berger [28]), which is defined as

$$H_o(\Omega) = \{v \in L^2(\Omega) \ ; \ \frac{1}{r} |\nabla v|^2 \in L^1(\Omega), \ v|_{\partial\Omega} = 0\}.$$

More precisely, $H_o(\Omega)$ is the closure of the standard test functions space $\mathcal{D}(\Omega)$ for the Hilbert norm

$$(8.1) \qquad ||v||_{H_o(\Omega)} \equiv \left(\int_\Omega |v|^2 dx + \int_\Omega \frac{1}{r} |\nabla v|^2 \, dx\right)^{1/2}.$$

Hence, we first look for a solution (u_a,W_a) (resp. (u_a,k_a) of

Problem (I_a) (three-dimensional) :

$$(I_a) \qquad \begin{cases} \mathcal{L}u = f(u - \frac{W}{2} r^2 - k) \text{ in } \Pi_a, \\ u \in H_o(\Pi_a), \\ \int_{\Pi_a} \frac{1}{r} |\nabla u|^2 \, dx = \eta. \end{cases}$$

Notice that the seminorm

$$(8.2) \qquad [\![v]\!] = \left(\int_{\Pi_a} \frac{1}{r} |\nabla v|^2 \, dx \right)^{1/2}$$

is actually a norm on $H_o(\Pi_a)$, equivalent to $\|\cdot\|_{H_o(\Omega)}$ (see [23]), whence

$$(8.3) \qquad \tilde{a}(u,v) = \int_{\Pi_a} \frac{1}{r} \nabla u \cdot \nabla v \, dx$$

is an $H_o(\Pi_a)$-elliptic symmetric bilinear form. Due to the particular form of the scalar product and the singularity at r=0, we are led to make a different choice of P_K. Thus, we further impose :

Property 1 : "There exists h_o : $0 < h_o \leq h$ such that all triangles in $\mathcal{T}_h$ having one vertex (resp. one side) on $\partial\Pi$, have one side (resp. one vertex) on the line $r = h_o$.

Let us set

$$(8.4) \qquad P_K = \begin{cases} P_\ell(K), \text{ if } K \cap \partial\Pi = \emptyset, \\ P_\ell^{r,\alpha}(K), \text{ otherwise} \end{cases}$$

where $\alpha > 3/2$, ℓ=1 or 2, and

$$P_\ell^{r,\alpha}(K) = \{q \; ; \; q = r^\alpha p, \text{ with } p \in P_\ell(K), \; p|_{K \cap \partial\Pi} \equiv 0\}.$$

Clearly,

$$V_h = \{v_h \in C^o(\overline{\Pi}_a) \; ; \; v_h|_K \in P_K \; \forall K \in \mathcal{T}_h, \; v_h = 0 \text{ on } \partial\Pi_a\}$$

is a finite-dimensional subspace of $H_o(\Pi_a)$. As a consequence of Property 1, a function $v_h \in V_h$ is completely determined by :

(a) Its values on the vertices of $\mathcal{T}_h$ belonging to Π_a, if $\ell=1$.

(b) Its values on the vertices and middle-points of the sides of $\mathcal{T}_h$ belonging to Π_a, if $\ell=2$.

For this choice of triangulations and basis functions, the finite element approximation can be formulated again, and the convergence procedure of Section 6 holds.

Remark 8.2 : Monotonicity properties for A_h can be easily obtained for the bi-dimensional discrete problems. Indeed, it suffices to apply the results of P.G. Ciarlet and P.A. Raviart [21], by virtue of which, if all angles of all triangles of $\mathcal{T}_h$ are $\leq \pi/2$, then A_h^{-1} has only non-negative elements. Under this condition, if $\mathcal{T}_h$ satisfies Property 1, and if any angle having its vertex but no side on $\partial\Pi$ is $<\pi/2$, then the matrix A_h corresponding to the three-dimensional problem is also of monotone class (see E. Fernandez Cara [27], for further details). ∎

9. A VARIABLE MESH-SIZE PROCEDURE.

In this Section we are concerned with the numerical solution of the unbounded vortex problem (I) and (II). As previously indicated, a classical finite element procedure consists in solving the corresponding Π_a-approximations for different values of a converging to $+\infty$. Indeed, if the solutions (u_a,k_a) of (II_a)-k (say) are such that the canonical extensions $\tilde{u}_a$ converge weakly in $H_0^1(\Pi)$ to a function u_*, and k_a converges to a real value $k_* > 0$, one can prove that (u_*,k_*) is a solution of (II)-k. In practice, it suffices to have the vorticity regions uniformly bounded and the functions u_a satisfying a Cauchy convergence test. However, this carries some computational difficulties : we must solve many times (I_a) (or (II_a)) to obtain a good information about the behaviour of a solution of (I) (resp. (II)), and, on the other hand, when a is "large", appropriate triangulations result in a considerable computational effort which may increase round-off errors.

For these reasons, a variable mesh-size method is adequate to solve the unbounded problem. We start from the following relations

$$(9.1)\quad \begin{cases} \displaystyle\iint_{\mathcal{O}} \nabla u\cdot\nabla v \; dr\, dz \\ \displaystyle\quad = \int_{-\pi/2}^{\pi/2}\int_{\rho_o}^{\rho_1} \rho\{\frac{\partial u}{\partial \rho}\frac{\partial v}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial u}{\partial \theta}\cdot\frac{\partial v}{\partial \theta}\}\, d\rho\, d\theta \\ \displaystyle\quad = \int_{-\pi/2}^{\pi/2}\int_{t_o}^{t_1} \{\frac{\partial u}{\partial t}\;\frac{\partial v}{\partial t} + \frac{\partial u}{\partial \theta}\;\frac{\partial v}{\partial \theta}\}\, dt\, d\theta, \end{cases}$$

valid for $\rho_i = e^{t_i}$ $(i=0,1)$,

$$\mathcal{O} = \{x \in \Pi;\; x = (r,z),\; \rho_o < |x| < \rho_1\} \;,$$

and for all $u,v \in H^1(\mathcal{O})$, by means of the relations

$$(9.2a)\qquad r = \rho \cos\theta,\; z = \rho \sin\theta$$

and

$$(9.2b)\qquad \rho = e^t.$$

Now we set

$$\hat{\mathcal{O}} = \{(t,\theta) \in \mathbb{R}^2 \;;\; t_o < t < t_1,\; |\theta| \le \pi/2\},$$

and we call $\hat{\nabla}$ the gradient operator in the variables t and θ. Then (9.1) reads

$$(9.3)\qquad \iint_{\mathcal{O}} \nabla u\cdot\nabla v \; dr\, dz = \iint_{\hat{\mathcal{O}}} \hat{\nabla} u\cdot\hat{\nabla} v\; dt\, d\theta.$$

Let us associate to each $a > 0$ the open set

$$D_a = \{x \in \Pi;\; x=(r,z),\; |x| < e^a\}.$$

Thus, if σ is a given fixed positive value and $a > \sigma$, we have the following splitting of D_a :

$$D_a = D_\sigma \cup \mathcal{O}_{\sigma,a} \cup Z_{\sigma,a} \;,$$

where

$$\mathcal{O}_{\sigma,a} = \{x \in \Pi;\ x=(r,z),\ e^{\sigma} < |x| < e^{a}\}$$

and

$$Z_{\sigma,a} = D_a \setminus (\mathcal{O}_{\sigma,a} \cup D_{\sigma}).$$

For a given positive integer s (to converge to $+\infty$), we define D_{σ}^{s} as the open region in Π bounded by the axis $r=0$ and all segments such that

$$S_i = [X_i, X_{i+1}],\ 0 \le i \le 2s-1, \tag{9.4a}$$

where

$$X_i = e^{\sigma}(\cos(\tfrac{\pi}{4}(\tfrac{i}{s}-1)),\ \sin(\tfrac{\pi}{4}(\tfrac{i}{s}-1))). \tag{9.4b}$$

Next we associate to each s a real parameter h in such a way that $h \to 0$ if and only if $s \to +\infty$ (for example, $h=1/s$), and to every h a triangulation $\mathcal{T}_{\sigma,h}$ of D_{σ}^{s}. We suppose that the family $\{\mathcal{T}_{\sigma,h}\}$ satisfies the following three properties :

(9.5) "<u>The only vertices of</u> $\mathcal{T}_{\sigma,h}$ <u>lying on</u> $\mathcal{J}_{\sigma,a}$ <u>are the</u> X_i's",

$$\lim_{h\to 0}\{\max_{K\in\mathcal{T}_{\sigma,h}} \delta(K)\} = 0, \tag{9.6}$$

(9.7) " $\exists C > 0$ <u>such that</u> $\delta(K)/\rho(K) \le C \quad \forall K \in \mathcal{T}_{\sigma,h},\ \forall h$"

Let us also associate to each s a triangulation $\hat{\mathcal{T}}_{\sigma,a,h}$ (in the (t,θ)-plane) of the open set

$$\hat{\mathcal{O}}_{\sigma,a} = \{(t,\theta) \in \mathbb{R}^2\ ;\ \sigma < t < a,\ |\theta| < \pi/2\}$$

for which the unique vertices with $t=\sigma$ are the images of the points X_i under the transformation (9.2), i.e., the points

$$\hat{X}_i = (\sigma, \tfrac{\pi}{4}(\tfrac{i}{s}-1)),\ 0 \le i \le 2s-1.$$

Next we introduce the finite dimensional spaces

$$(9.8)_h \quad \begin{cases} V_{\sigma,h} = \{v_h \in C^0(\bar{D}^s_\sigma) \; ; \; v_h|_K \in P_1(K), \quad \forall K \in \mathcal{T}_{\sigma,h} \; ; \\ v_h = 0 \; \underline{\text{on}} \; \partial\Pi \cap \partial D^s_\sigma\} \, , \end{cases}$$

$$(9.9)_h \quad \begin{cases} \hat{V}_{\sigma,a,h} = \{\hat{v}_h \in C^0(\bar{\hat{\mathcal{O}}}_{\sigma,a}) \; ; \; \hat{v}_h|_{\hat{K}} \in P_1(\hat{K}), \; \forall \hat{K} \in \hat{\mathcal{T}}_{\sigma,a,h} \; ; \\ v_h = 0 \; \underline{\text{on}} \; \partial\hat{\mathcal{O}}_{\sigma,a} \cap \partial\hat{D}_a\} \, . \end{cases}$$

Let $w^1_{\sigma,h},\dots,w^m_{\sigma,h}$ (resp. $\hat{w}^1_{\sigma,a,h},\dots,\hat{w}^{\hat{m}}_{\sigma,a,h}$) be the usual basis functions of the space $V_{\sigma,h}$ (resp. $\hat{V}_{\sigma,a,h}$), and let us set $w^j_{\sigma,a,h}$ for the inverse transform in the (r,z)-plane of $\hat{w}^j_{\sigma,a,h}$:

$$\hat{w}^j_{\sigma,a,h}(t,\theta) \equiv w^j_{\sigma,a,h}(e^t\cdot\cos\theta, e^t\cdot\sin\theta).$$

We finally consider the space V_h generated by <u>all functions</u> $w^i_{\sigma,h}$ and $w^j_{\sigma,a,h}$. Each $v_h \in V_h$ is clearly continuous on $D^s_\sigma \cup \mathcal{O}_{\sigma,a}$ (and in general discontinuous on D_a), we are led to a non-conforming approximation. This can be avoided by using isoperimetric finite elements (see e.g. P.G. Ciarlet [20]). The resulting discrete formulations of problem (II) read now : <u>Find a function</u> $u_h \in V_h$ <u>and a real parameter</u> $W_h > 0$ (<u>resp.</u> $k_h \geq 0$), <u>solutions of</u> :

$$(9.10)_h \quad \begin{cases} \displaystyle\iint_{D^s_\sigma} \nabla u_h\cdot\nabla v_h \, dr\, dz + \int_{-\pi/2}^{\pi/2}\int_\sigma^a \hat{\nabla} u_h\cdot\hat{\nabla} v_h \, dt\, d\theta \\ \displaystyle = \iint_{D^s_\sigma} f(u_h - Wr - k)v_h \, dr\, dz + \int_{-\pi/2}^{\pi/2}\int_\sigma^a f(u_h - We^t\cos\theta - k)v_h e^{2t} dt\, d\theta, \\ \forall v_h \in V_h \, , \\ \displaystyle\iint_{D^s_\sigma} |\nabla u_h|^2 \, dr\, dz + \int_{-\pi/2}^{\pi/2}\int_\sigma^a |\hat{\nabla} u_h|^2 \, dt\, d\theta = \eta. \end{cases}$$

One can deal analogously with the three-dimensional problems (I), but the resulting equations are somewhat more complicated :

$$(9.11)_h \begin{cases} \displaystyle\iint_{D_\sigma^s} \frac{1}{r} \nabla u_h \cdot \nabla v_h \, dr\, dz + \int_{-\pi/2}^{\pi/2}\int_\sigma^a \frac{1}{e^t \cos\theta} \hat{\nabla} u_h \cdot \hat{\nabla} v_h \, dt\, d\theta \\ \displaystyle = \iint_{D_\sigma^s} f(u_h - \frac{W}{2} r^2 - k) v_h \, dr\, dz + \int_{-\pi/2}^{\pi/2}\int_\sigma^a f(u_h - \frac{W}{2} e^{2t} \cos^2\theta - k) v_h \, e^{2t} dt\, d\theta, \\ \forall v_h \in \tilde{V}_h, \\ \displaystyle\iint_{D_\sigma^s} \frac{1}{r} |\nabla u_h|^2 \, dr\, dz + \int_{-\pi/2}^{\pi/2}\int_\sigma^a \frac{1}{e^t \cos\theta} |\hat{\nabla} u_h|^2 \, dt\, d\theta = \eta. \end{cases}$$

In $(9.11)_h$, $\tilde{V}_h$ is slightly modified from V_h, and constructed as in Section 8 by replacing those $P_1(K)$ for which $K \cap \partial\Pi \neq \emptyset$ by $P_\ell^{r,\alpha}(K)$, with $\alpha > 3/2$.

This method has the advantage of simplifying the computational work considerably : since the exponential function increases rapidly, for "moderate" values of a we are concerned with "large" domains. The stiffness matrices are analogous, since the scalar products

$$\int \nabla u \cdot \nabla v \, dx \quad \text{and} \quad \int \frac{1}{r} \nabla u \cdot \nabla v \, dx$$

are preserved after applying (9.2).

Remark 9.1 : An alternative strategy in dealing with the above difficulties consists in introducing a conformal mapping from $\mathcal{O}_{\sigma,\infty} \equiv \Pi \setminus \bar{D}_\sigma$ onto D_σ and solving the unbounded discrete problems themselves. This procedure is studied in E. Fernandez Cara [27]. ∎

10. NUMERICAL EXPERIMENTS AND FURTHER COMMENTS.

This Section deals with some numerical experiments obtained by applying the methods discussed previously. The computations illustrate the results and conjectures of Section 3 and clearly exhibit typical vortex structures.

The problems have been solved separately by several algorithms, but only results achieved by (A.1) will be presented here since this method seemed to be the most performing one. We have used piecewise linear finite elements to approximate the vortex pair equations and the special elements described in Section 8 (with $\ell=1$, $\alpha=2$) for the three-dimensional case. In all experiments the convergence test has been

$$\frac{\sum_i |u_i^{n+1}-u_i^n|}{\sum_i |u_i^{n+1}|} ,$$

where we have taken $\varepsilon = 10^{-5}$, 10^{-6} or 10^{-7}. The equations

$$\alpha(u^n,\lambda) = \eta$$

(see Section 7) have been solved by means of a dichotomy procedure, which is appropriate here because of the monotonicity of $\alpha(u^n,\cdot)$.

All computations were made at I.N.R.I.A. (Rocquencourt, France), on Multics CII-Honeywell System, and for mesh generation a MODULEF's procedure was used.

10.1. <u>First examples of vortex pair problems</u>.

We present some results concerning vortex pair problems in various <u>bounded domains</u>. We recall that these problems read :

Given the energy parameter $\eta > 0$, the vorticity function f and $W > 0$ (resp. $k \geq 0$), to find a function u and a real value k (resp. W) satisfying :

$$(10.1) \quad \begin{cases} -\Delta u = f(u-Wr-k), \ x = (r,z) \in \Omega, \\ u \in H_0^1(\Omega), \\ \int_\Omega |\nabla u|^2 \, dx = \eta . \end{cases}$$

Here, Ω is defined to be either

$$(10.2) \qquad \Omega \equiv R(a) = (0,a) \times (-a,a),$$

or

$$(10.3) \qquad \Omega \equiv B(a) = \{x \in \mathbb{R}^2 \ ; \ x = (r,z), \ |x|<a, r > 0\},$$

for large $a > 0$. Both cases are intended to provide an approximation to the unbounded domain problem.

By a result of B. Gidas, Wei-Ming Ni and L. Niremberg [30] (see also M.J. Esteban [24]), the solutions are Steiner-symmetric with respect to the axis z=0. Hence, for numerical purposes, it will suffice to solve the problem in the half-region

$$\Omega_+ = \{x \in \Omega \ ; \ x = (r,z), \ z > 0\} \tag{10.4}$$

with imposing homogeneous Neumann's boundary conditions on the boundary

$$\Gamma_o = \{x \in \overline{\Omega} \ ; \ x = (r,z), \ z=0\} \ .$$

Let us first take Ω as in (10.2). Our aim is to let an increase towards infinity and to check whether the computed solutions approach the solution of any vorticity problem. A good numerical test can be obtained by using identity (3.13). Notice that, for a bounded domain $\Omega = R(a)$, (3.13) changes into

$$\gamma \equiv Wr_c\chi - 2\mu = -\frac{1}{2}\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 x\cdot n \, d\sigma \ , \tag{10.5}$$

with n being the outward unit normal on $\partial\Omega$. Indeed,

$$\nabla\cdot z \equiv \nabla\cdot((r \frac{\partial u}{\partial r} + z \frac{\partial u}{\partial z})\nabla u - \frac{1}{2} \ |\nabla u|^2 x)$$

$$= (r \frac{\partial u}{\partial r} + z \frac{\partial u}{\partial z})\Delta u \ ,$$

whence

$$\int_\Omega f(u\text{-}Wr\text{-}k)(r \frac{\partial u}{\partial z} + z \frac{\partial u}{\partial z})dx = -\int_{\partial\Omega} z\cdot n \, d\sigma \ .$$

Thus, setting $U = u\text{-}Wr\text{-}k$ we obtain

$$\nabla\cdot(F(U)x) = (r \frac{\partial U}{\partial r} + z \frac{\partial U}{\partial z})f(U) + 2F(U),$$

while

$$r \frac{\partial U}{\partial r} + z \frac{\partial U}{\partial z} = r \frac{\partial u}{\partial r} + z \frac{\partial u}{\partial z} - Wr,$$

and (10.5) is proved.

<u>Remark 10.1</u> : From (10.5), one has

$$Wr_c\chi \leq 2\mu \tag{10.6a}$$

for a bounded domain $\Omega = R(a)$. Similar arguments lead to

$$-W(a-r_c)\chi = 2\mu - \frac{1}{2}\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 (r-a,z)\cdot n\, d\sigma ,$$

which in turn yields :

(10.6b) $\quad -W(a-r_c)\chi \leq 2\mu$. ∎

As a first example, we have chosen for a special vorticity function :

(10.7) $\quad f(s) = \lambda s_+,$

where $\lambda > 0$. Some examples of triangulations are given in Fig. 1 to 5. The approximate solution of (10.1) has been computed for certain values of the data, and a visualisation of the flow (the streamlines

$$\psi \equiv u-Wr-k = \text{constant},$$

the shape of the vorticity core regions

$$A_\psi = \{x \in \Omega \ ; \ x = (r,z), \ \psi(x) > 0\}$$

and the velocity fields) is given in Fig. 6 to 15.

It is readily seen that the stream function ψ agrees very well with the picture of a vortex. That is, there exists a bounded (small) region determined by the existence of a vorticity motion, while the velocity field tends to a constant vector (vertically oriented upwards, as in Section 2) at infinity.

An evaluation of functionals μ, r_c, χ and

$$\gamma = Wr_c\chi - 2\mu$$

is given in Table 1 for different values of a and fixed W. Their values strongly support the convergence of the approximation as $a \uparrow +\infty$.

For large a (see Fig. 12 to 16) the computed vorticity region is small and near-circular. In the limit $a = +\infty$, some asymptotics are known for the solution provided it exists (see L. Lichenstein [34,35], H. Lamb [33], P.G. Saffman [45]) :

$$W \sim \frac{\chi}{4\pi r_c} , \quad k \sim \frac{\eta}{\chi} \tag{10.10}$$

Thus, for given λ, k, η with large k/η, one has :

$$\chi \sim \frac{\eta}{k} , \quad W r_c \sim \frac{\eta}{4\pi k} . \tag{10.11}$$

Also, using the near-circularity of A_ψ (see J. Norbury [43]), one obtains

$$\lambda \sim \frac{j_o^2}{\rho^2} , \tag{10.12}$$

with ρ being the ratio of A_ψ and $j_o = 2.402$, so that

$$\eta \sim \frac{\chi^2}{2\pi} \log \frac{4\lambda r_c^2}{j_o^2} , \tag{10.13}$$

$$r_c \sim \frac{j_o}{2} \lambda^{-\frac{1}{2}} \exp \{\frac{\pi}{\chi^2}\eta\} , \tag{10.14}$$

$$W \sim \frac{2}{j_o} (\frac{\eta}{4\pi k}) \lambda^{1/2} \exp \{- \frac{2\pi k^2}{\eta}\} . \tag{10.15}$$

These approximate identities are checked agains our results in Table 2. Again the adequation is quite satisfactory.

In the previous computations we have remarked that the unknowns W and k are seen numerically to be in a functional relation. That is, a given W leads (e.g. via (A.1)) to a value of k that itself leads back (via (A.1)) to the initial W. This strongly suggests that there exists a function

$$\zeta : [0,+\infty) \to \mathbb{R}_+$$

such that

$$W = \zeta(k), \quad k = \zeta^{-1}(W) ;$$

the conjectures in Section 3 just mean that ζ is continuous and decreasing and satisfies

$$\zeta(0) = W_2^*, \quad \lim_{k \uparrow +\infty} \zeta(k) \downarrow 0.$$

10.2. Some numerical experiments with other vorticity functions.

The vortex pair problem (10.1) with

$$(10.17)\qquad f(s) = H(s),\ H(s) = \begin{cases} 1 \ \underline{\text{for}}\ s > 0, \\ 0 \ \underline{\text{otherwise}}, \end{cases}$$

is of particular importance (see e.g. B. Turkington [51], R.T. Pierrehumbert [43]). However, the discontinuity at the origin requires techniques which are different from those used in this paper, and we will rather consider functions f as in Remark 4.2 :

$$(10.18)\qquad f(s) = \begin{cases} (\frac{s}{1+\beta})^{\beta} \ \underline{\text{for}}\ s > 0,\ 0 < \beta \leq 1, \\ 0 \ \underline{\text{for}}\ s \leq 0. \end{cases}$$

$$(10.19)\qquad f(s) = \begin{cases} 1+\beta(s-\varepsilon) \ \underline{\text{for}}\ s > \varepsilon,\ 0 < \beta \leq \varepsilon \leq 1, \\ s/\varepsilon \ \underline{\text{for}}\ 0 < s \leq \varepsilon, \\ 0 \ \underline{\text{for}}\ s \leq 0. \end{cases}$$

Indeed, for small values of β and ε, these vorticity functions provide a good first approximation of the Heaviside function (10.17).

Fig. 16 to 30 deal with some experiments concerning the solution of (10.1) where f is given by (10.18) or (10.19). The numerical results have been checked against (3.13) for large values of a (see Tables 3 and 4).

Notice that for large a (see Fig. 23, 24, 29 and 30) the vorticity core region becomes again small and near-circular. In the case f(s) = H(s) and a = + ∞, certain asymptotics can be derived using (10.10) and (10.11) (see [42]) :

$$(10.20)\qquad \eta \sim \frac{\chi^2}{4\pi} \log \frac{4\pi r_c^2}{\chi} ,$$

$$(10.21)\qquad W \sim (\frac{\eta}{4\pi k})^{1/2} \exp\{- \frac{2\pi k^2}{\eta}\} .$$

These are checked against our results in Tables 5 and 6.

10.3. Some experiments concerning three-dimensional problems.

Let us now turn to some numerical results concerning (three-dimensional) vortex ring problems in bounded domains. We recall that these problems read :

Given $\eta > 0$, the vorticity function f and $W > 0$ (resp. $k \geq 0$), to find a function u and a real value k (resp. W) satisfying :

$$(10.22)\quad \begin{cases} \mathcal{L}u \equiv -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial u}{\partial r}\right) - \dfrac{1}{r}\dfrac{\partial^2 u}{\partial z^2} = f(r, u - \dfrac{W}{2} r^2 - k),\ x = (r,z) \in \Omega, \\ u \in L^2(\Omega),\ \dfrac{1}{r}\ |\nabla u|^2 \in L^1(\Omega),\ u=0 \ \underline{\text{on}}\ \partial\Omega, \\ \displaystyle\int_\Omega \frac{1}{r}\,|\nabla u|^2\, dx = \eta\ . \end{cases}$$

The vorticity function f has been chosen as follows :

$$(10.23)\qquad f(r,s) = \lambda r \cdot s_+\ ,$$

with $\lambda > 0$ being a constant.

For domains $\Omega = R(a)$ (cf. (10.2)), we have solved approximately (10.22) for certain values of the data. The computed streamlines

$$\psi(x) \equiv u(x) - \frac{W}{2} r^2 - k = \text{constant}$$

and vorticity regions

$$A_\psi = \{x \in \Omega\ ;\ x = (r,z),\ \psi(x) > 0\}$$

are displayed in Fig. 31 to 33.

10.4. An analysis of the computed vorticity regions.

For a clearer presentation, let us consider problems (10.1) with $\Omega = R(10.)$, $\eta = 20.$, f as in (10.7) (with λ=1.), n=20 and W taking all values in Table 7. The corresponding computed streamlines and vorticity (upper half-) regions are displayed in Fig. 34 to 45.

We have remarked the existence of a well-defined symmetry (with respect to W=0) of the shapes of the vorticity regions that can be explained by means of the change r' = a-r. Indeed, if (u_a, k_a) is a solution of (10.1) corresponding to a non positive value W_o of W, the function v_a, defined as

$$(10.24)\qquad v_a(r,z) = u_a(a-r,z),$$

satisfies

$$-\Delta v_a(r,z) = f(v_a(r,z)-(-W_o)r - (k_a+aW_o))$$

for all $(r,z) \in \Omega$, and

$$\int_\Omega |\nabla v_a|^2\, dx = \int_\Omega |\nabla u|^2\, dx = \eta .$$

In other words, (v_a, k_a+aW_o) is a solution of (10.1) corresponding to the value $-W_o$ of the vortex velocity parameter. The above mentioned property is now a consequence of the fact that the vorticity regions corresponding to $(u_a,k_a;W_o)$ and $(v_a,k_a+aW_o;-W_o)$ are symmetrically situated with respect to the axis a=r/z.

For $W < 0$, the vorticity region is never bounded idenpendently of a, for it has a nonempty intersection with the half-plane $r > a/2$. From a strictly numerical viewpoint this shows that (II)-k has no solution for $W < 0$ (which is easy derived numerically). When W=0 (see B. Gidas ; W.M. Ni and L. Nirenberg [30]), the solution u_a and the function v_a, defined as in (10.24), must coincide, i.e., the vorticity region is symmetric with respect to both the r=a/2 and z=0 axis. This is confirmed by the shape in Fig. 34. Then, for small positive values of the velocity (see Fig. 35 to 43), the pair lies between the W=0 and the cylindrical vortex (for which k=0 ; see below) ; this means that the vorticity region remains bounded. Hence we are led to a "critical" value W^* of W, in such a way that the corresponding value of k is zero. In fact, numerical results show that $W < W^*$ if and only if $k > 0$, and this is a necessary condition for having the existence of a solution of (II)-k. These remarks strongly support Conjecture 2 mentioned in section 1.

For $W > W^*$ (Fig. 44 and 45), we find again unbounded vorticity regions. This shows numerically (as in the case $W < 0$) that (II)-k has no solution (see Theorem 2 in Section 3). However, the solutions that we have computed for which $W > W^*$ satisfy a property of weak convergence to zero in $H^1_o(\Pi)$. This can be seen, by looking at Fig. 46, in which the relationship between W and k is represented for different values of a. We wish to emphasize that this feature of the bounded problems is precisely what makes difficult the attack of Conjecture 3 by a limiting process.

The behavior of the three-dimensional vorticity regions, as well as their interpretation, is quite similar.

10.5. An approximation of the cylindrical vortex.

Let us now consider more precisely those two-dimensional vortex problems for which $\Omega = B(a)$, f is given by (10.7) with λ=1. and k is close to zero. We recall that for k=0 an explicit solution of (10.1) in B(a) is known, namely

$$u \begin{cases} W\rho \cos\theta - \dfrac{2W}{J_o(\alpha)} J_1(\rho) \cos\theta, & \rho^2 \equiv r^2+z^2 \leq \alpha^2, \\ \dfrac{W^2}{a^2-\alpha^2} \left(\dfrac{a^2}{\rho} - \rho\right)\cos\theta, & \alpha \leq \rho \leq a. \end{cases}$$

This is the cylindrical vortex (see J. Norbury [40]), for which

$$W^2 = \eta/(2\pi\alpha^2),$$

where $\alpha = 3.831\ldots$ is the first zero of the Bessel's function of the first kind J_1.

For $\varepsilon=10^{-6}$, taking refined meshes in a neighbourhood of the vorticity region A_ψ (see the variable mesh-size method in Section 9), we solved approximately problem (10.1) using the following data :

(10.25) $\Omega = B(100.)$, $\eta = 1000.$, f <u>is given by</u> (10.7) <u>with</u> $\lambda=1.$,

(10.26) $W = 3.2, 3.225, 3.25, 3.29305.$

The shapes of the corresponding vorticity half-regions are displayed in Fig. 47 to 50. There we use triangulations of maximum diameter equal to 1.5, .5 and .2 (in a neighbourhood of A_ψ), resulting in the regions bounded by the z=0 axis and the curves (1),(2) and (3), resp. In order to improve the convergence of (A.1), we have introduced a relaxation parameter $\omega = 1.11$. For $W = 3.29305$ (see Fig. 50) we have, up to an error $< 10^{-6}$,

$$\eta \simeq 2\pi\cdot W^2\cdot\alpha^2,$$

and the corresponding value of k is in all cases very close to zero. This clearly shows that our method yields quite a good approximation of the cylindrical vortex. Indeed, in this case, the vorticity half-region

$$\{x \in \mathbb{R}^2 ; x=(r,z), r,z > 0, r^2+z^2 < \alpha^2\}$$

practically coincides with $A_\psi^+ = A_\psi \cap \Omega^+$.

10.6. The asymptotic relationships between the vorticity velocity and the flux parameters.

All computations in this subsection have been carried out with a 10^{-5} precision. We already know that for every non negative value of k there exists a solution of the two-dimensional problem (II)-W. On the other hand (see Theorem 2 in Section 3), (II)-k has no solution when the velocity parameter W is chosen $>W_2^*$, for a certain positive W_2^*. From a numerical viewpoint (see Section 9), one considers (10.1)-W and (10.1)-k with Ω = R(a) and a large enough. This should yield a good approximation of the corresponding unbounded problems. Thus, a relationship between W and k (associated to a and becoming stable as $a \to +\infty$) can be established, by means of which we can describe the behaviour of different vortex pairs.

The very same considerations stand for (three-dimensional) vortex rings.

Fig. 52 corresponds to the solution of (10.1) where Ω = R(50.), f is given by (10.7) with λ=2. and η = 1000. An analogous three-dimensional example is displayed in Fig. 53, where Ω = B(100.), f is as in (10.23) with λ=2. and η =1000.

a	k	χ	r_c	μ	γ
50.	3.7352	5.1855	7.2018	.9853	-.1031
100.	3.7951	5.1923	8.1512	1.0892	-.0622
200.	3.8200	5.1970	10.1287	1.0767	-.0215
400.	3.8391	5.2006	10.1305	1.3210	-.0088
800.	3.8461	5.2010	10.1303	1.3174	-.0004

Table 1

Evaluation of the unknown k and the functionals χ, r_c, μ and $\gamma = Wr_c\chi - 2\mu$ corresponding to the computed solution of (10.1) where $\Omega = R(a)$, f is given by (10.7) with $\lambda=1.$, $\eta=20.$ and $W=.05$ is prescribed.

a	k	W'	k'	η'	r'_c	W"
50.	3.7352	.0573	3.8569	15.33124	10.5591	.0467
100.	3.7951	.0507	3.8518	17.8315	10.4947	.0428
200.	3.8200	.0408	3.8484	19.7312	10.4506	.0413
400.	3.8391	.0409	3.8457	19.7600	10.4170	.0401
800.	3.8461	.0409	3.8454	19.7629	10.4133	.0397

Table 2

Evaluation of the unknown k and the expressions $W' = \chi/(4\pi r_c)$, $k' = \eta/\chi$, $\eta' = \chi^2/2\pi \log(4\lambda r_c^2/j_o^2)$, $r'_c = j_o/2\ \lambda^{-1/2} \cdot \exp(\pi\eta/\chi^2)$, $W'' = 2\eta/(4\pi k j_o)\lambda^{1/2} \cdot \exp(-\pi k^2/\eta)$, corresponding to the computed solution of (10.1) where $\Omega = R(a)$, f is given by (10.7) with λ=1., η=20. and W=.05 is prescribed.

a	k	χ	r_c	μ	γ
50.	1.3212	14.1033	3.1516	15.9834	-.8533
100.	1.3484	14.3821	1.7791	8.4225	-.0121
200.	1.3548	14.7601	1.8570	9.5938	-.0010

Table 3

Evaluation of the unknown k and the functionals χ, r_c, μ and $\gamma = W r_c \chi - 2\mu$ corresponding to the computed solution of (10.1) where $\Omega = R(a)$, f is given by (10.18) with β=.01, η=20. and W=.7 is prescribed.

a	k	χ	r_c	μ	γ
50.	1.3014	14.0201	3.1814	16.1107	-.9989
100.	1.3328	14.2937	1.8034	9.5513	-.0586
200.	1.3547	14.7598	1.8835	9.7311	-.0015

Table 4

Evaluation of the unknown k and the functionals χ, r_c, μ and $\gamma = Wr_c\chi - 2\mu$ corresponding to the computed solution of (10.1) where $\Omega = R(a)$, f is given by (10.19) with β=.01 and ε=.05, η=20. and W=.7 is prescribed.

a	k	W'	k'	η'	r_c'	W"
50.	1.3212	.3561	1.4181	34.5123	1.9927	.6342
100.	1.3484	.6433	1.3906	16.7442	1.9640	.6503
200.	1.3548	.6925	1.3550	18.6721	1.9295	.6689

Table 5

Evaluation of the unknown k and the functionals $W' = \chi/(4\pi r_c)$, $k' = \eta/\chi$, $\eta' = \chi^2/4\pi \cdot \log(4\pi r_c^2/\chi)$, $r_c' = (\chi/\pi)^{1/2}/2 \cdot \exp(2\pi\eta/\chi^2)$, $W'' = (\eta/4\pi k)^{1/2} \cdot \exp(-2\pi k^2/\eta)$ corresponding to the computed solution of (10.1) where $\Omega = R(a)$, f is given by (10.18) with β=.01, η=20. and W=.7 is prescribed.

a	k	W'	k'	η'	r_c'	W''
50.	1.3014	.3507	1.4265	68.9865	2.0065	.6495
100.	1.3328	.6307	1.3992	34.1608	1.9728	.5005
200.	1.3547	.6736	1.3550	28.3259	1.9296	.6090

Table 6

Evaluation of the unknown k and the functionals
$W' = \chi/(4\pi r_c)$, $k'=\eta/\chi$, $\eta' = \chi^2/2\pi \cdot \log(4\pi r_c^2/x)$,
$r_c' = (\chi/\pi)^{1/2}/2 \cdot \exp(2\pi\eta/\chi^2)$, $W'' = (\eta/4\pi k)^{1/2} \cdot \exp(-2\pi k^2/\eta)$
corresponding to the computed solution of (10.1) where $\Omega = R(a)$, f is given by (10.19) with β=.01 and ε=.05, η=20. and W=.7 is prescribed.

W	k
0.	6.3716
.0001	6.1584
.001	5.8312
.01	4.2118
.1	3.5275
.2	3.0418
.25	2.6395
.3	2.1028
.35	1.5047
.4	.9920
.45	.1233
.465725	$1.38 \cdot 10^{-4}$
.5	-.0283
.6	-.0994

Table 7

Evaluation of the unknowns k corresponding to the computed solutions of prolems (10.1) where Ω = R(50.), f is given by (10.7) with λ=1., η=20. and W is prescribed.

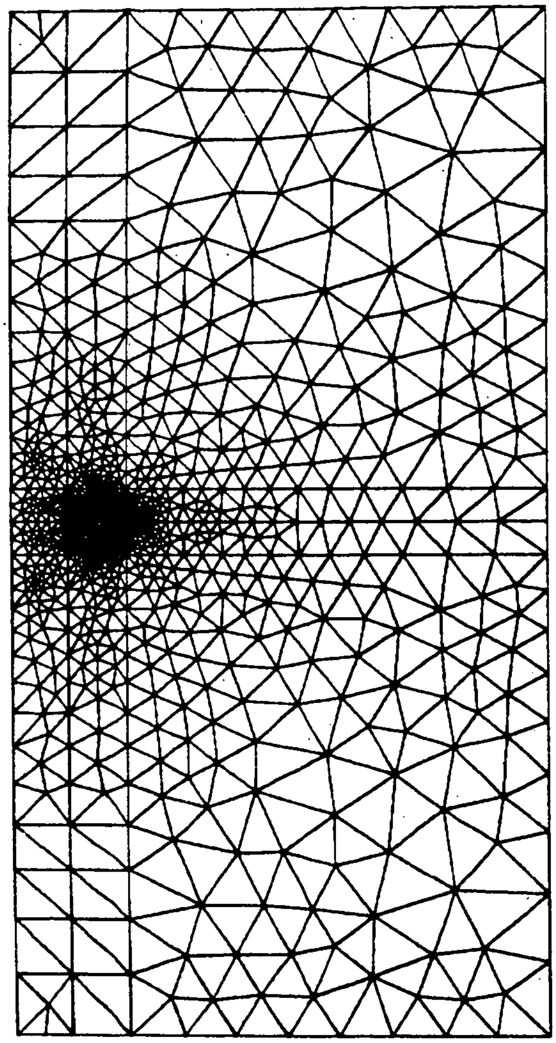


Fig. 1 : A first example of triangulation of the domain Ω = R(50.). Number of triangles : 1450. Number of nodal points : 763.

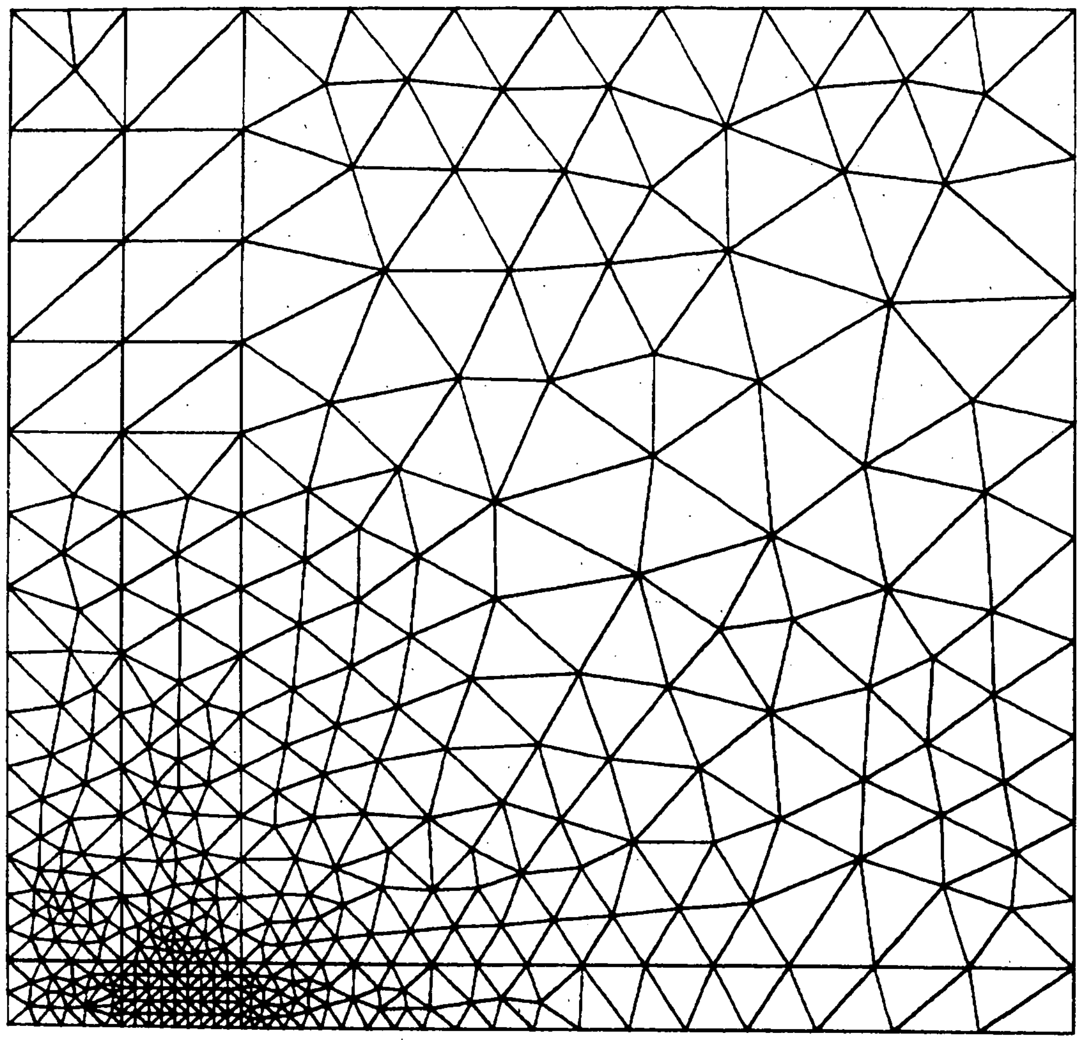


Fig. 2 : Detail of the triangulation in Fig. 1 : The upper subdomain $\Omega^+ = (0.,50.)\times(0.,50.)$. Number of triangles in Ω^+ : 725. Number of nodal points : 399. The solution of problem (10.1) was obtained by introducing an homogeneous Neumann's condition on $\Gamma_o = (0.,50.)\times\{0.\}$.

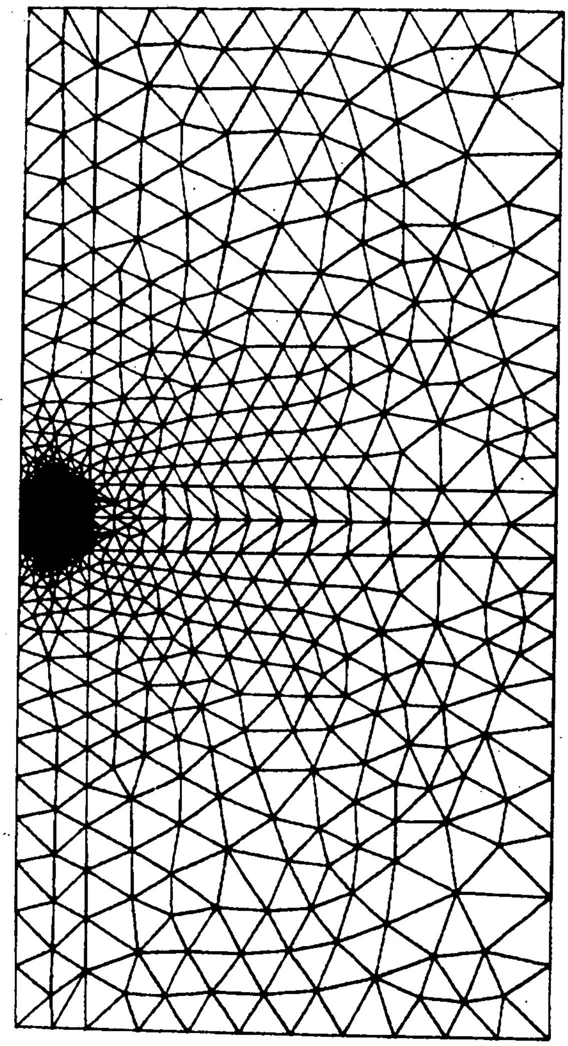


Fig. 3 : A second example of triangulation of the domain $\Omega = R(50.)$. Number of triangles : 1380. Number of nodal points : 732.

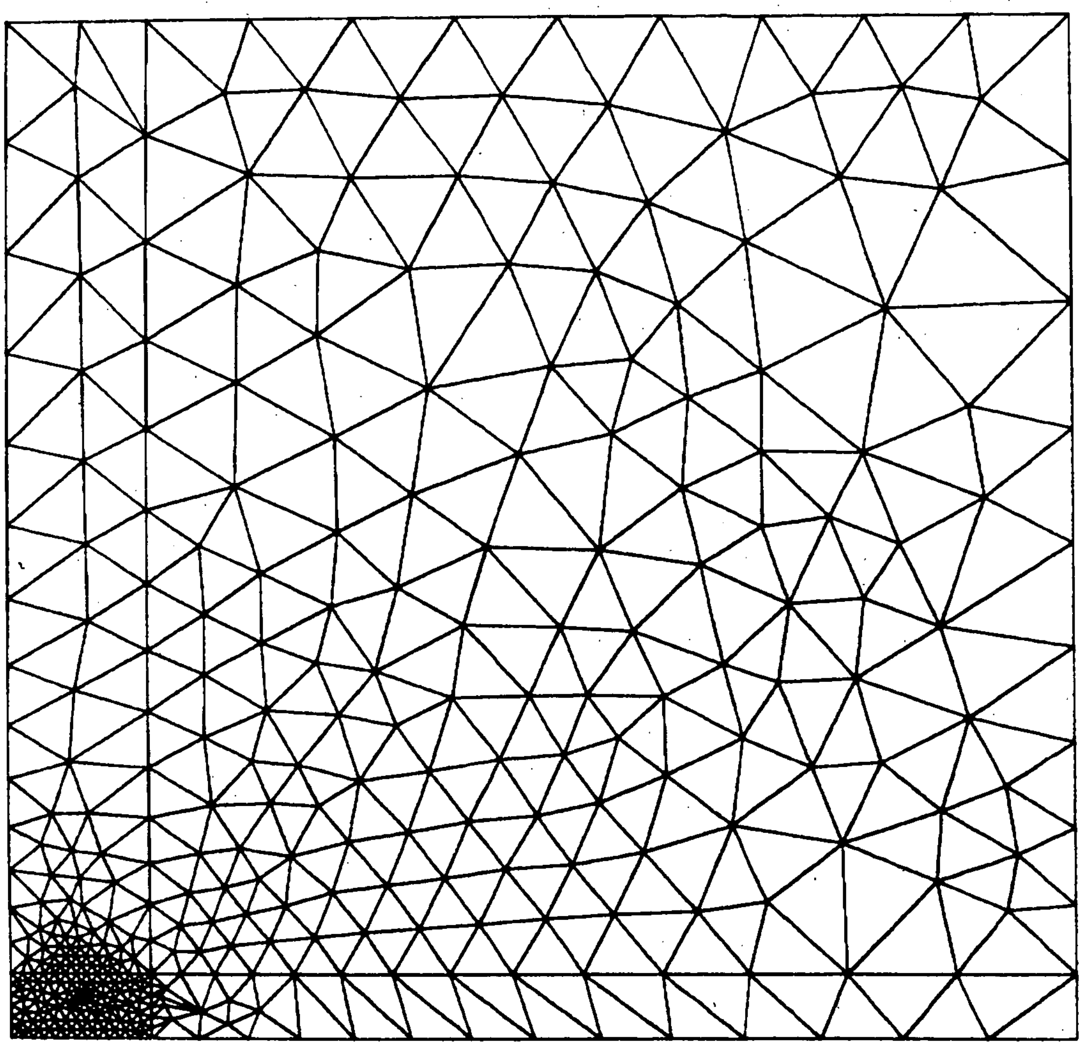


Fig. 4 : Detail of the triangulation in Fig. 3 : the upper subdomain $\Omega^+ = (0.,50.)\times(0.,50.)$. Number of triangles in Ω^+ : 690. Number of nodal points : 380. The solution of problem (10.1) was obtained by introducing an homogeneous Neumann's condition on $\Gamma_o = (0.,50.)\times\{0.\}$.

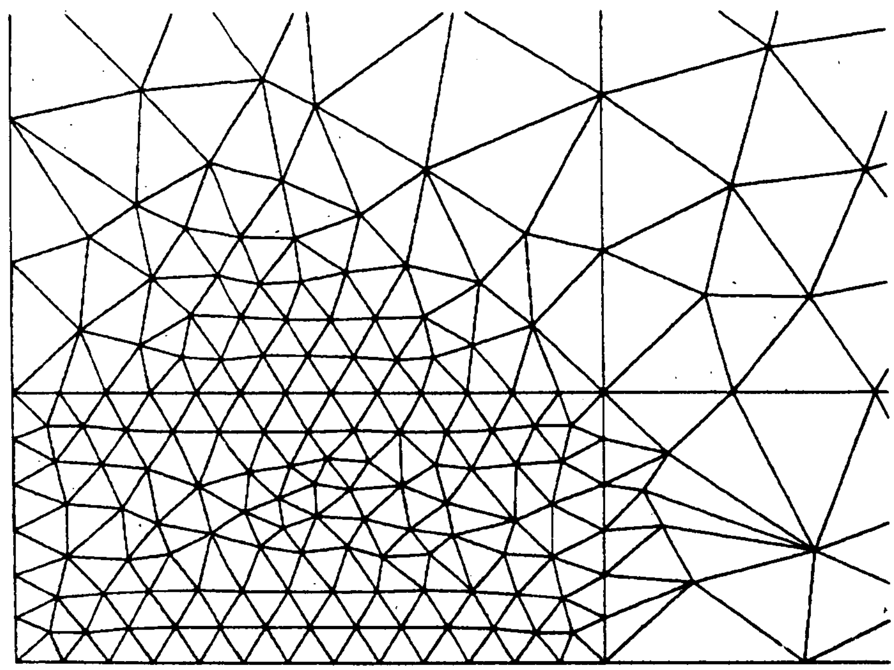


Fig. 5 : Detail of the (half-) triangulation in Fig. 4 : the subrectangle (0.,5.6)×(0.,4.)

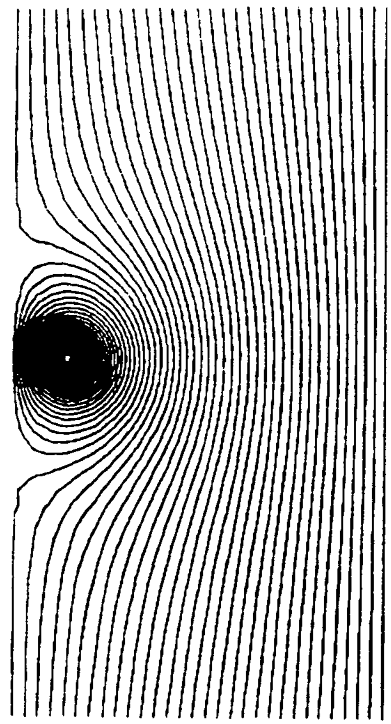


Fig. 6 : Streamlines $\psi \equiv u-Wr-k$ = constant and vorticity region $A_\psi = \{x \in \Omega \; ; \; x = (r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.7) with λ=1., η=20. and W=.05 is prescribed. The computed value of k is 3.7352. The triangulation is displayed in Fig. 1.

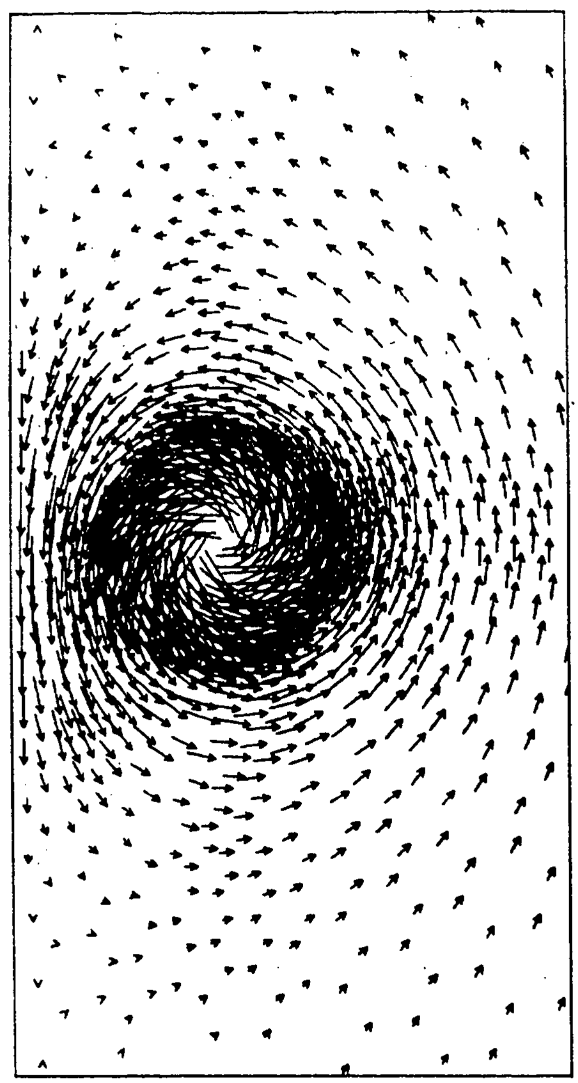


Fig. 7 : Detail of the computed velocity field corresponding to the stream function in Fig. 6. The field is displayed in the subdomain R(20.).

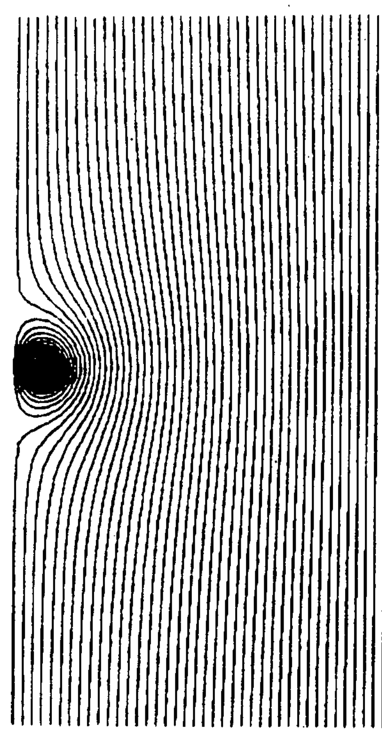


Fig. 8. : Streamlines $\psi \equiv u-Wr-k$ = constant and vorticity core region $A_\psi = \{x \in \Omega \ ; \ x=(r,z), \ \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(100.)$, f is given by (10.7) with λ=1., η=20. and W=.05 is prescribed. The computed value of k is 3.7951.

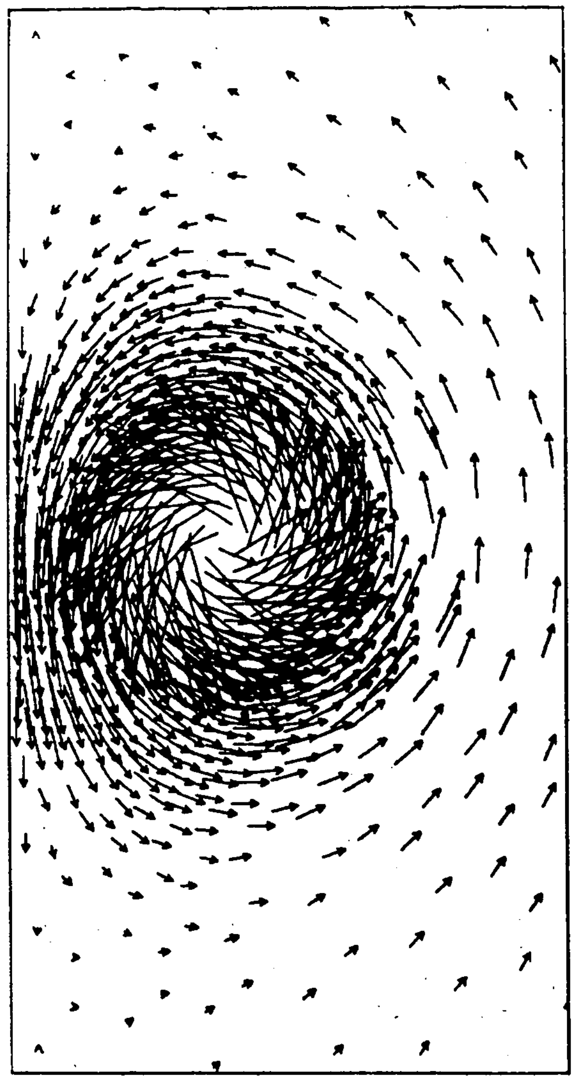


Fig. 9 : Detail of the computed velocity field corresponding to the stream function in Fig. 8. The field is displayed in the subdomain R (20.).

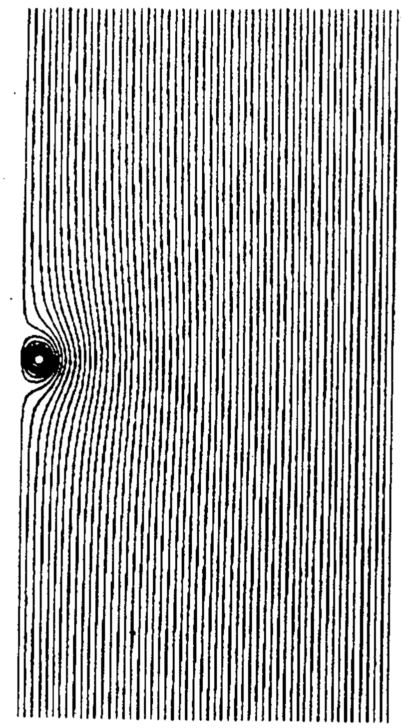


Fig. 10 : Streamlines $\psi \equiv u\text{-}Wr\text{-}k$ = constant and vorticity region $A_\psi = \{x \in \Omega \; ; \; x = (r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(200.)$, f is given by (10.7 with λ=1., η=20. and W=.05 is prescribed. The computed value of k is 3.8200.

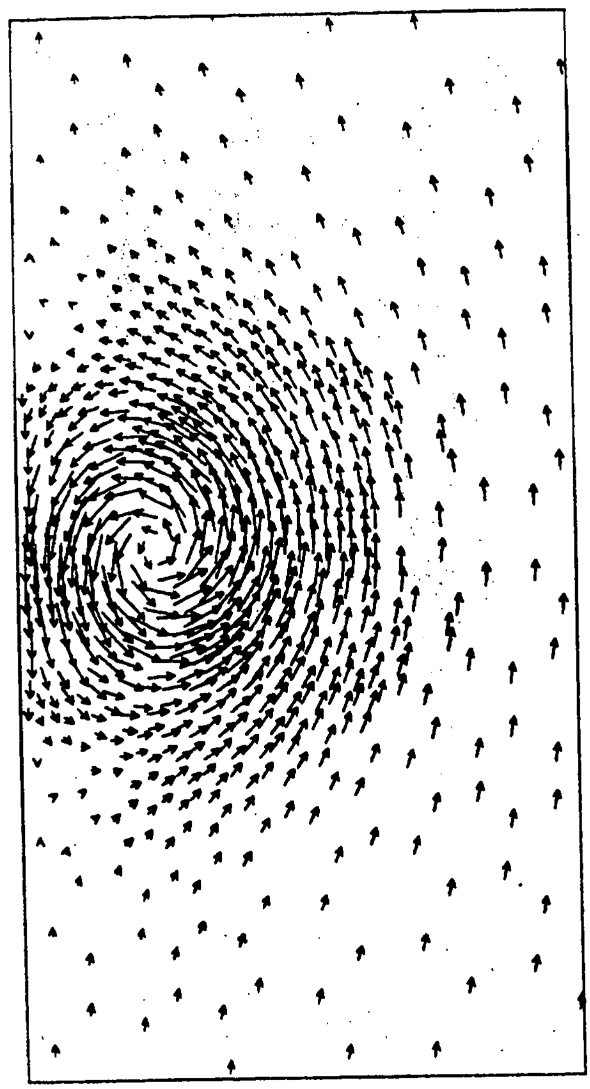


Fig. 11 : Detail of the computed velocity field corresponding to the stream function in Fig. 10. The field is displayed in the subdomain R(40.).

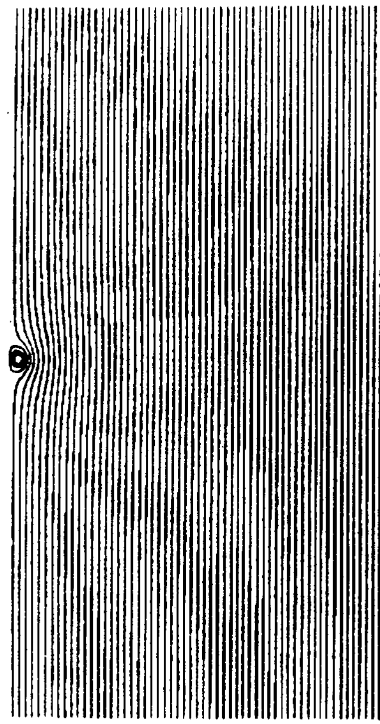


Fig. 12 : Streamlines $\psi \equiv u-Wr-k$ = constant and vorticity region $A_\psi = \{x \in \Omega \; ; \; x=(r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(400.)$, f is given by (10.7) with λ=1., η=20. and W=.05 is prescribed. The computed value of k is 3.8391.

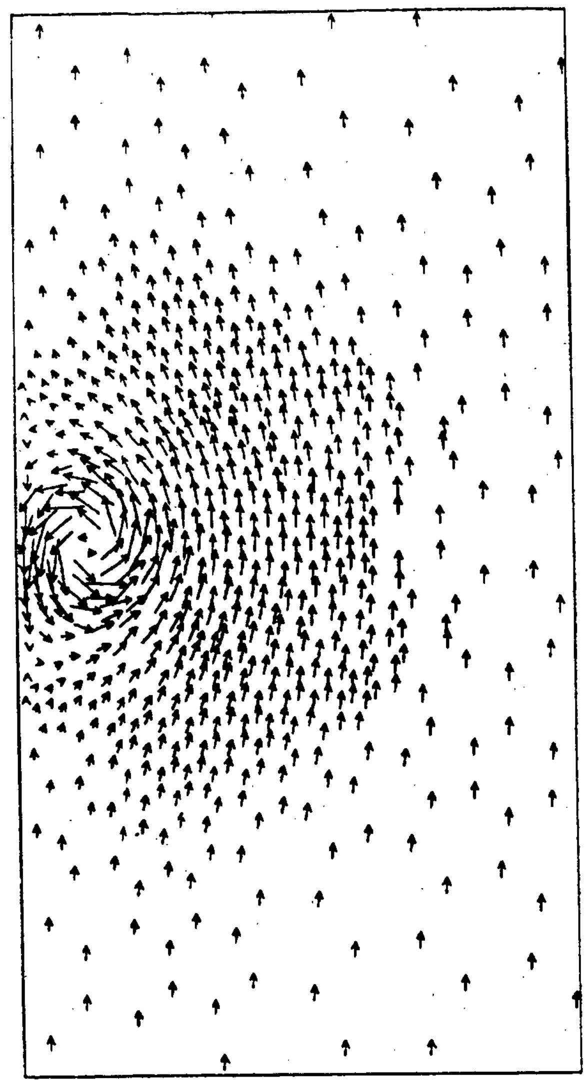


Fig. 13 : Detail of the computed velocity field corresponding to the stream function in Fig. 12. The field is displayed in the subdomain R (80.).

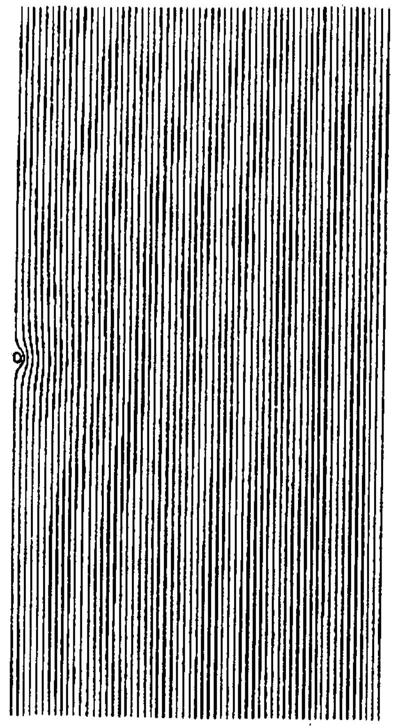


Fig. 14 : Streamlines $\psi \equiv u\text{-}Wr\text{-}k$ = constant and vorticity region $A_\psi = \{x \in \Omega \; ; \; x=(r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(800.)$, f is given by (10.7) with λ=1., η=20. and W=.05 is prescribed. The computed value of k is 3.8461.

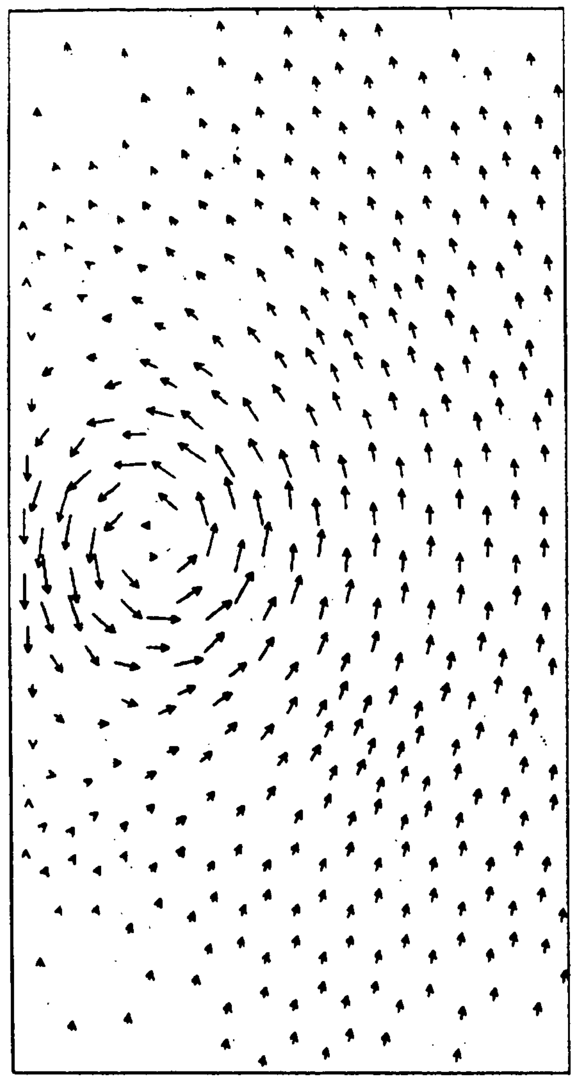


Fig. 15 : Detail of the computed velocity field corresponding to the stream function in Fig. 14. The field is displayed in the subdomain R(40.).

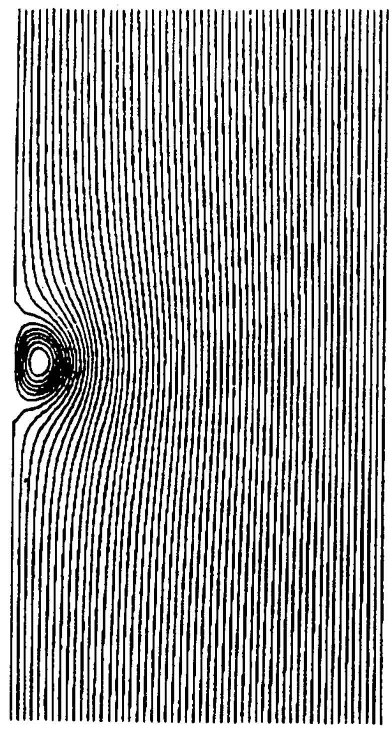


Fig. 16 : Streamlines $\psi \equiv u-Wr-k$ = constant and vorticity region $A_\psi = \{x \quad \Omega \; ; \; x = (r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.18) with $\beta = .01$, $\eta=20.$ and $W=.7$ is prescribed. The computed value of k is 1.3212

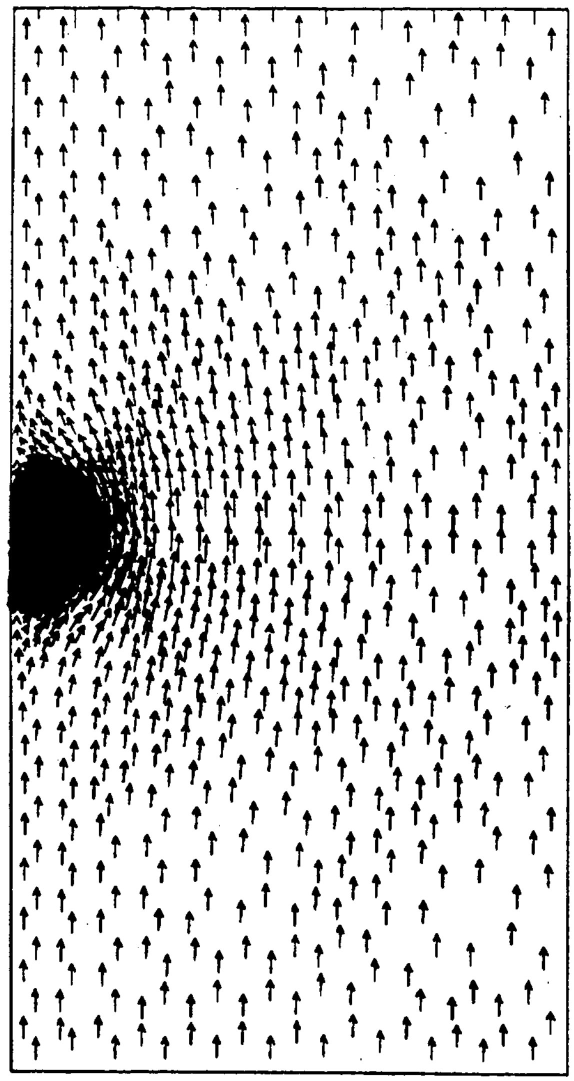


Fig. 17 : Velocity field corresponding to the stream function in Fig. 16. The field is displayed in the whole domain R(50.).

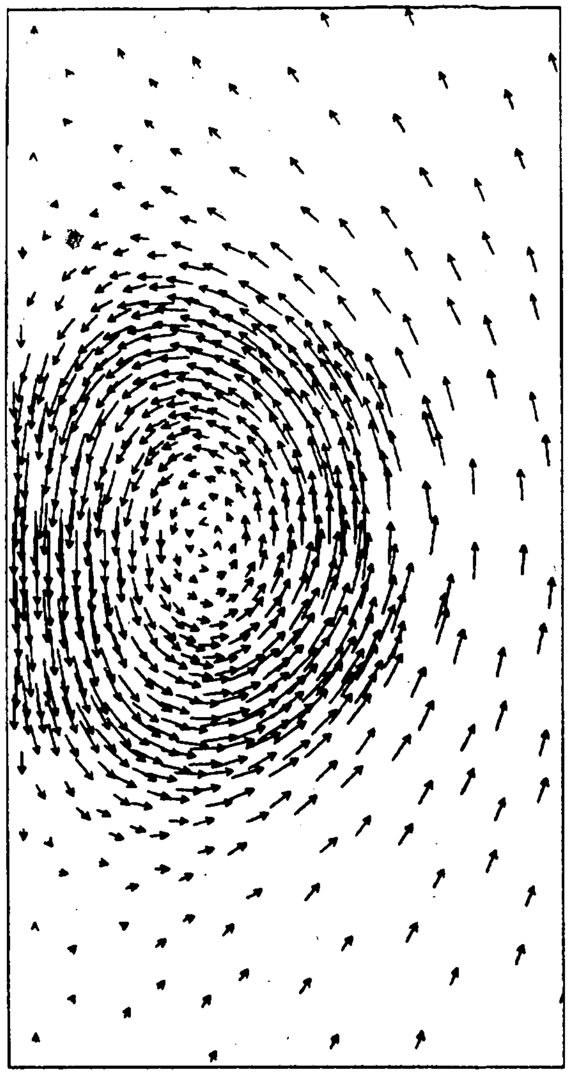


Fig. 18 : Detail of the velocity field corresponding to the stream function in Fig. 16. The field is displayed in the subdomain R(10.).

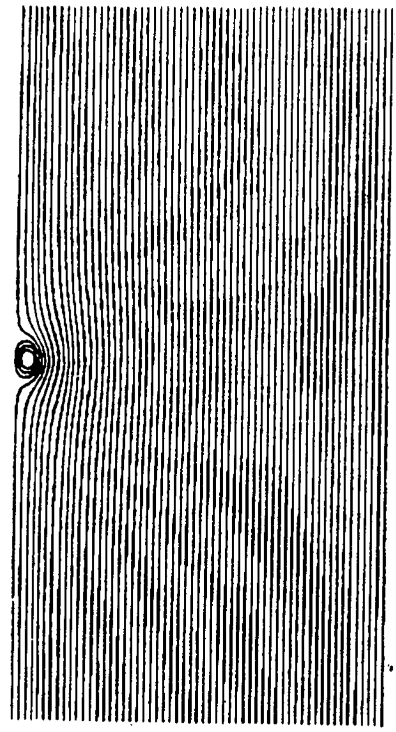


Fig. 19 : Streamlines ψ = u-Wr-k = constant and vorticity region $A_\psi = \{x \quad \Omega ; x=(r,z), \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where Ω = R(100.), f is given by (10.18) with β = .01, η=20. and W=.7 is prescribed. The computed value of k is 1.3484.

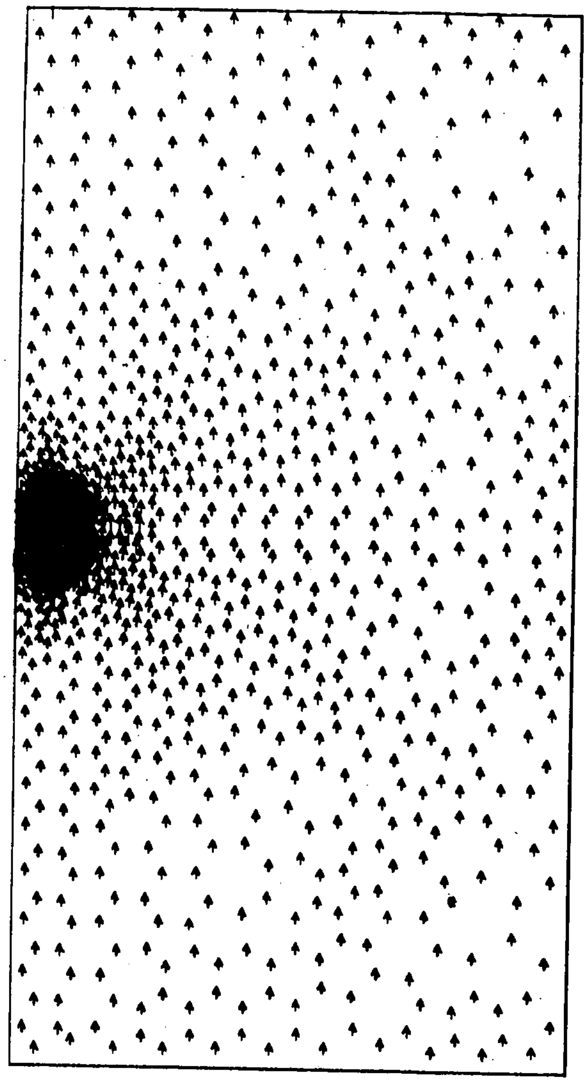


Fig. 20 : Velocity field corresponding to the stream function in Fig. 19. The field is displayed in the whole domain R(100.).

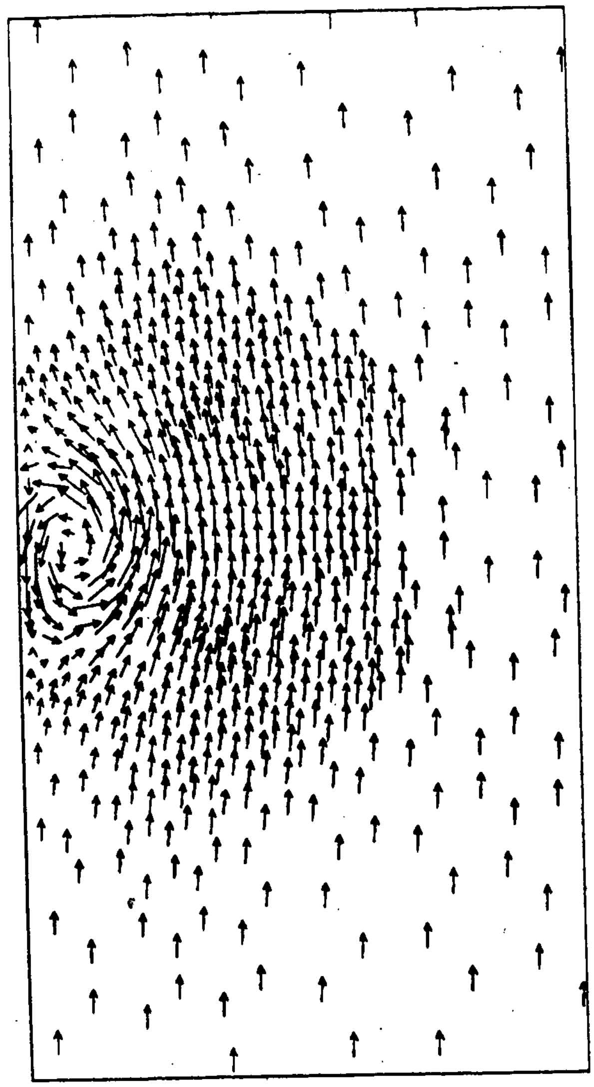


Fig. 21 : Detail of the vélocity field corresponding to the stream function in Fig. 19. The field is displayed in the subdomain R(20.).

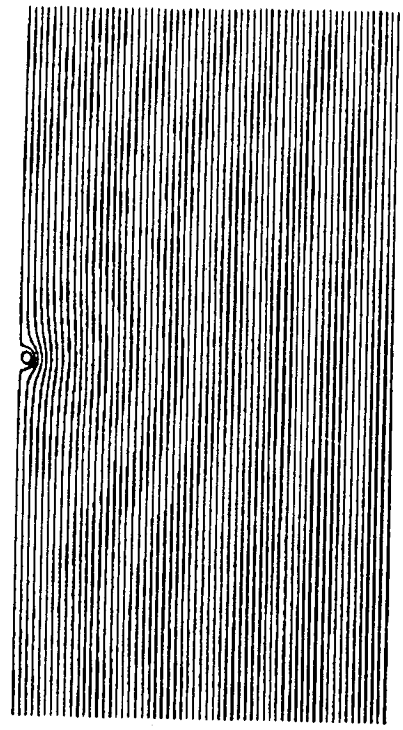


Fig. 22 : Streamlines $\psi \equiv u\text{-}Wr\text{-}k$ = constant and vorticity region $A_\psi = \{x \in \Omega \; ; \; x=(r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(200.)$, f is given by (10.18) with $\beta = .01$, $\eta = 20.$ and W=.7 is prescribed. The computed value of k is 1.3548.

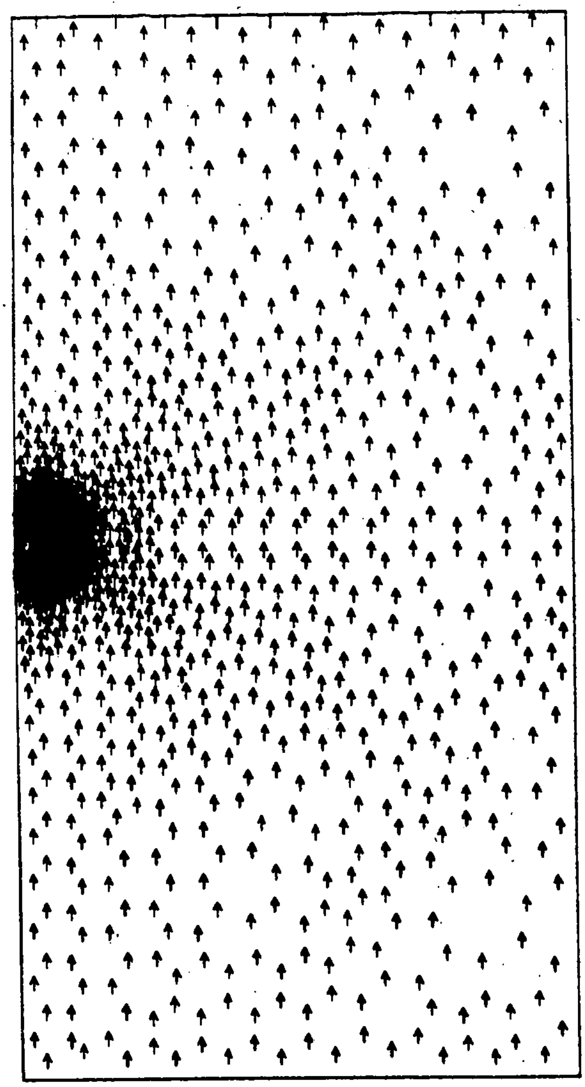


Fig. 23 : Velocity field corresponding to the stream function in Fig. 22. The field is displayed in the whole domain R(200.).

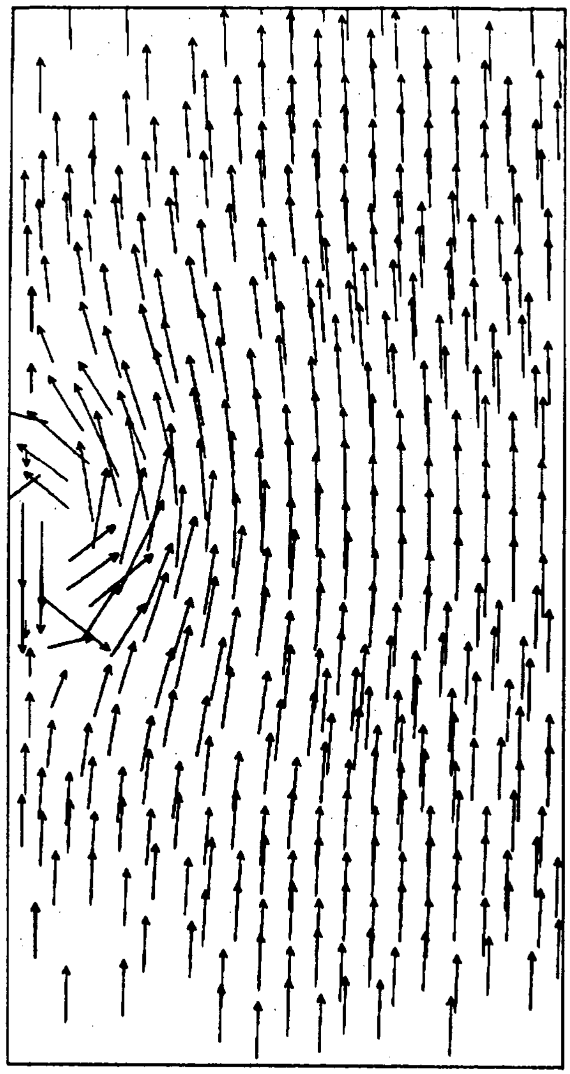


Fig. 24 : Detail of the velocity field corresponding to the stream function in Fig. 22. The field is displayed in the subdomain R(20.).

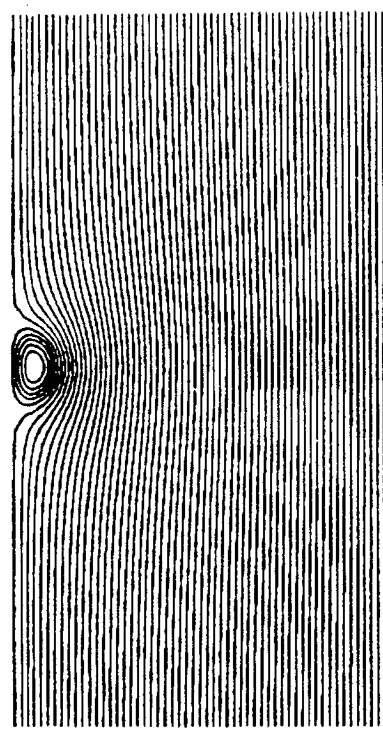


Fig. 25 : Streamlines $\psi \equiv u-Wr-k$ = constant and vorticity region $A_\psi = \{x \in \Omega \ ; \ x=(r,z), \ \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(5 \ )$, f is given by (10.19) with $\beta = .01$ and $\varepsilon=.05$, $\eta=20.$ and $W=.7$ is prescribed. The computed value of k is 1.3014.

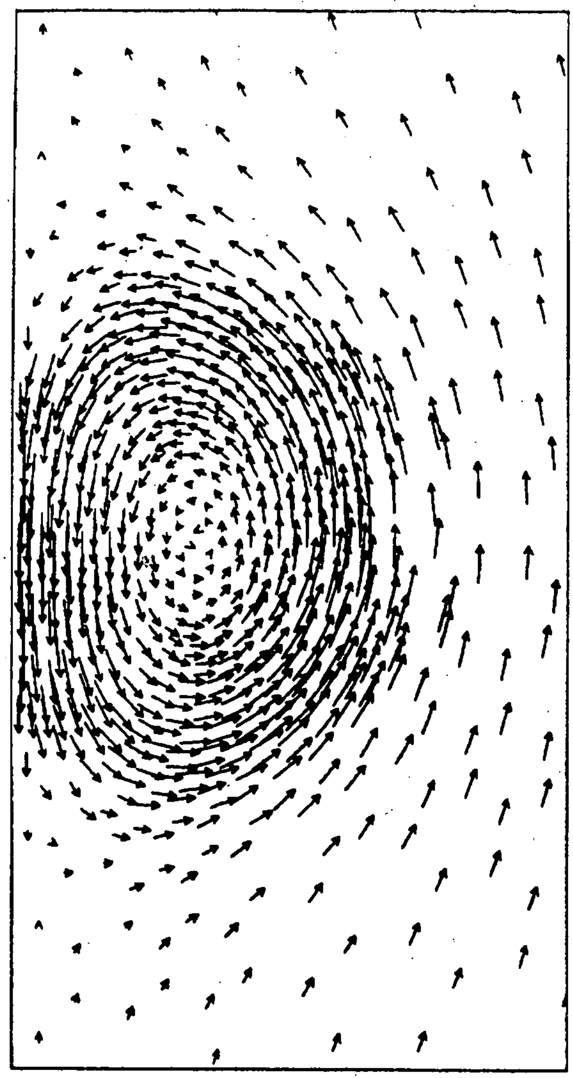


Fig. 26 : Detail of the velocity field corresponding to the stream function in Fig. 25. The field is displayed in the subdomain R(10.).

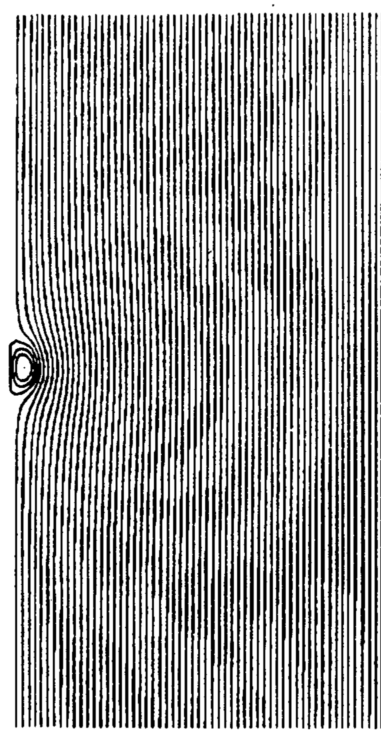


Fig. 27 : Streamlines $\psi \equiv u-Wr-k$ = constant and vorticity regions $A_\psi = \{x \in \Omega \ ; \ x=(r,z), \ \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(100.)$, f is given by (10.19) with $\beta = .01$ and $\varepsilon = .05$, $\eta = 20.$ and $W = .7$ is prescribed. The computed value of k is 1.3328.

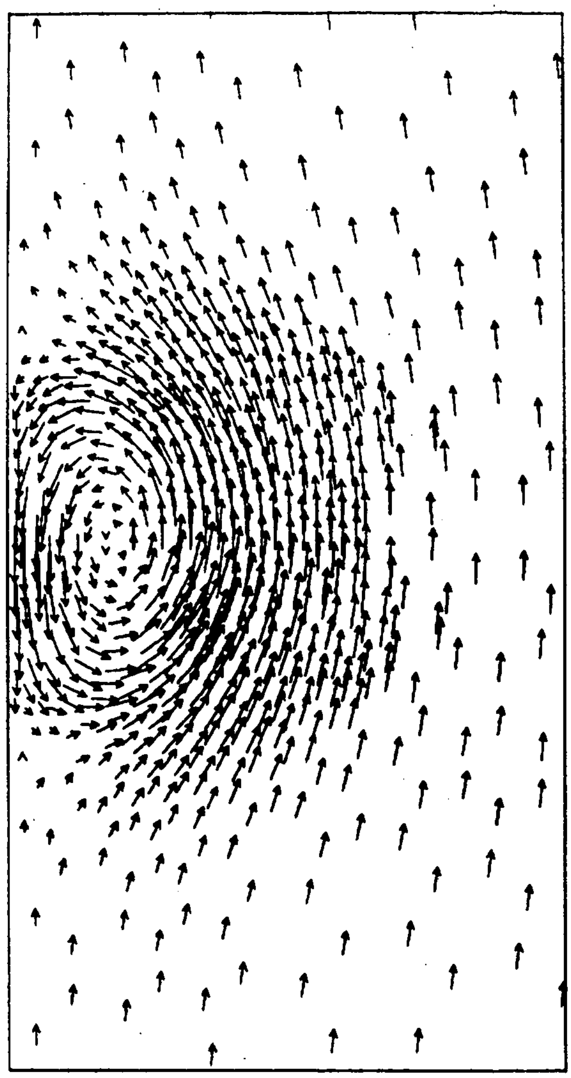
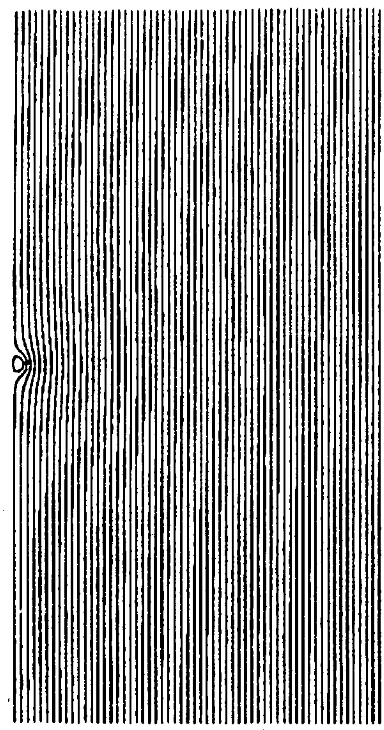


Fig. 28 : Detail of the velocity field corresponding to the stream function in Fig. 27. The field is displayed in the subdomain R(10.).



<u>Fig. 29</u> : Streamlines $\psi \equiv u-Wr-k$ = constant and vorticity region $A_\psi = \{x \in \Omega \; ; \; x=(r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where Ω = R(200., f is given by (10.19) with β=.01 and ε=.05, η=20. and W=.7 is prescribed. The computed value of k is 1.3547.

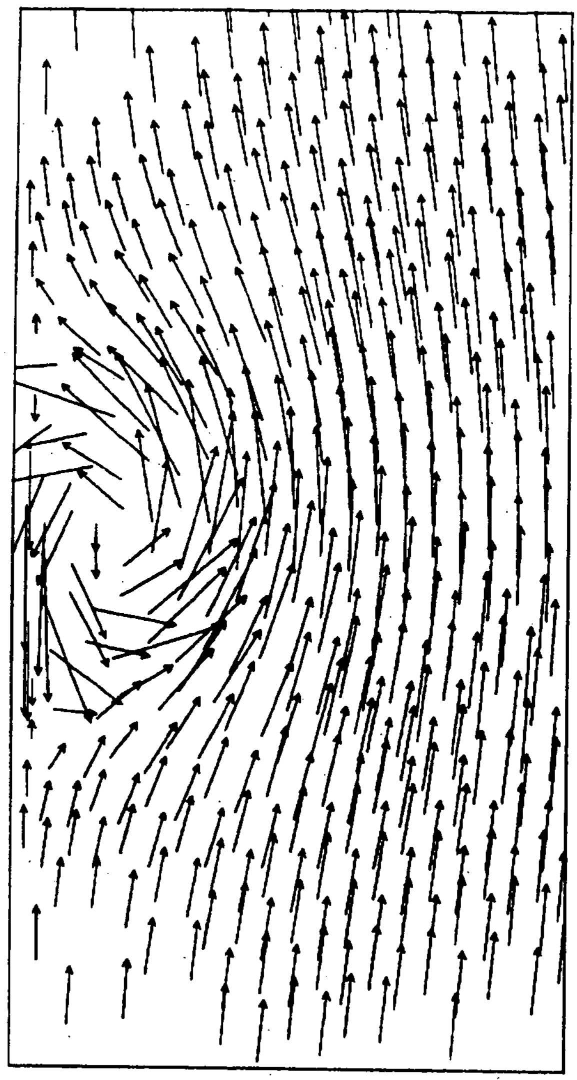


Fig. 30 : Detail of the velocity field corresponding to the stream function in Fig. 29. The field is displayed in the subdomain R(10.).

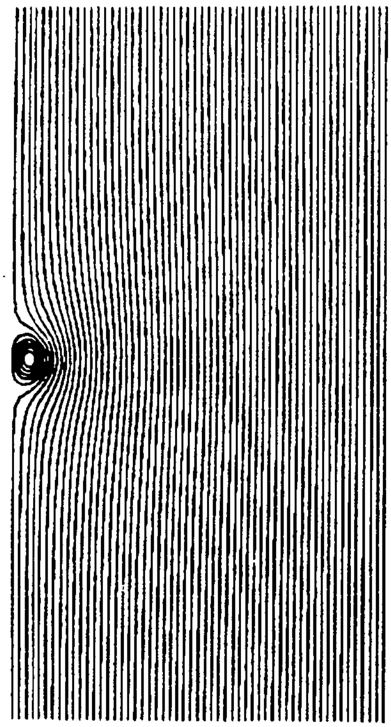


Fig. 31 : Streamlines $\psi \equiv u - \frac{W}{2} r^2 - k$ = constant and vorticity region $A_\psi = \{x \in \Omega \; ; \; x=(r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.22) where Ω=R(50.), f is given by (10.23) with λ=.1, η=20. and W=.01 is prescribed. The computed value of k is 3.5424.

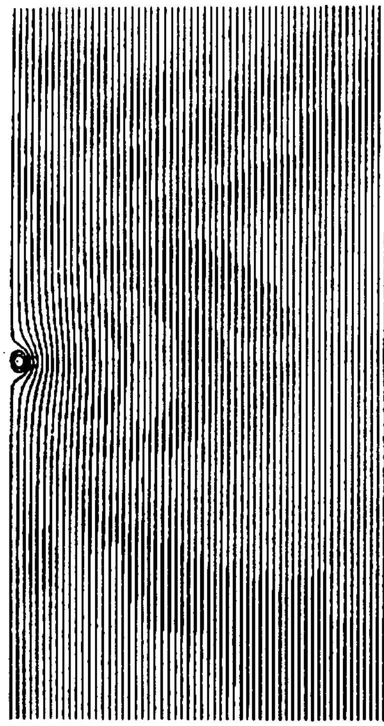


Fig. 32 : Streamlines $\psi \equiv u - \frac{W}{2} r^2 - k$ = constant and vorticity region $A_\psi = \{x \in \Omega \; ; \; x=(r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.22) where $\Omega = R(100.)$, f is given by (10.23) with λ=1., η=20. and W=.01 is prescribed. The computed value of k is 3.5612.

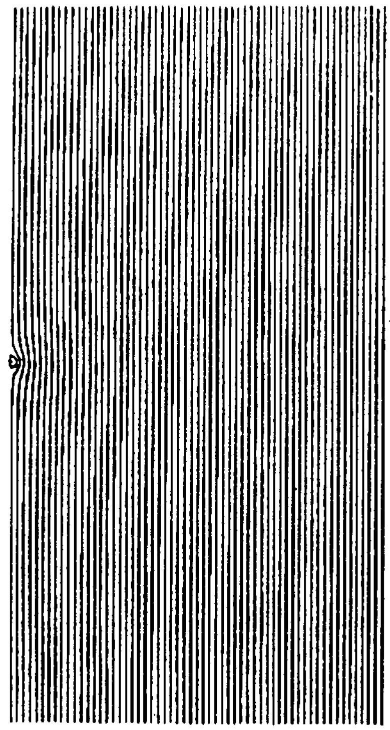


Fig. 33 : Streamlines $\psi \equiv u-\frac{W}{2}r^2-k$ = constant and vorticity region $A_\psi = \{x \in \Omega\ ;\ x=(r,z),\ \psi(x) > 0\}$ corresponding to the computed solution of (10.22) where $\Omega = R(200.)$, f is given by (10.23) with λ=.1, η=20. and W=.01 is prescribed. The computed value of k is 3.5650.

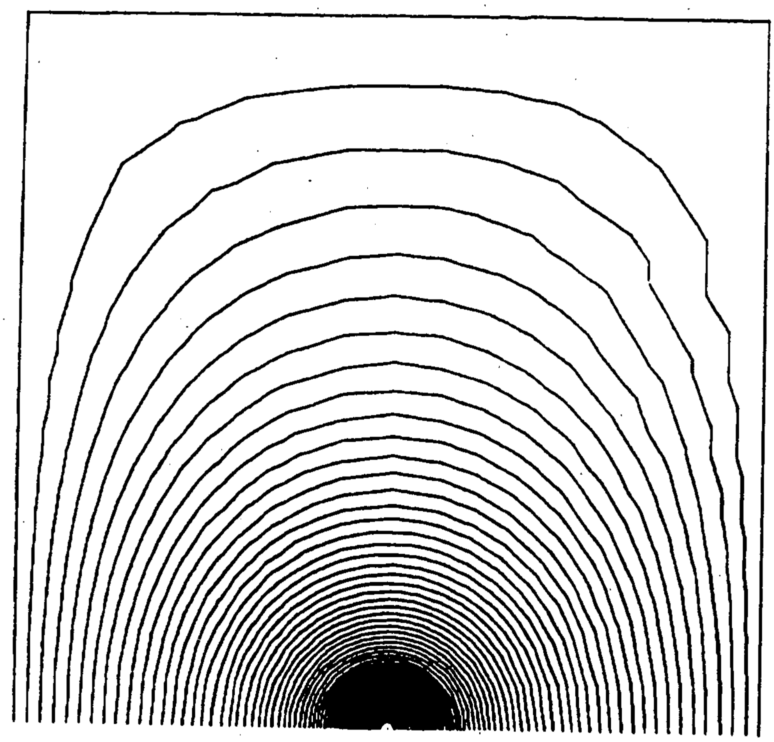


Fig. 34 : Streamlines $\psi \equiv u-Wr-k$ = constant and upper vorticity half-region $A_\psi^+ = \{x \in \Omega^+ \text{ ; } x=(r,z),\ \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.7) with λ=1., η=20. and W=0. is prescribed. The computed value of k is 6.3716.

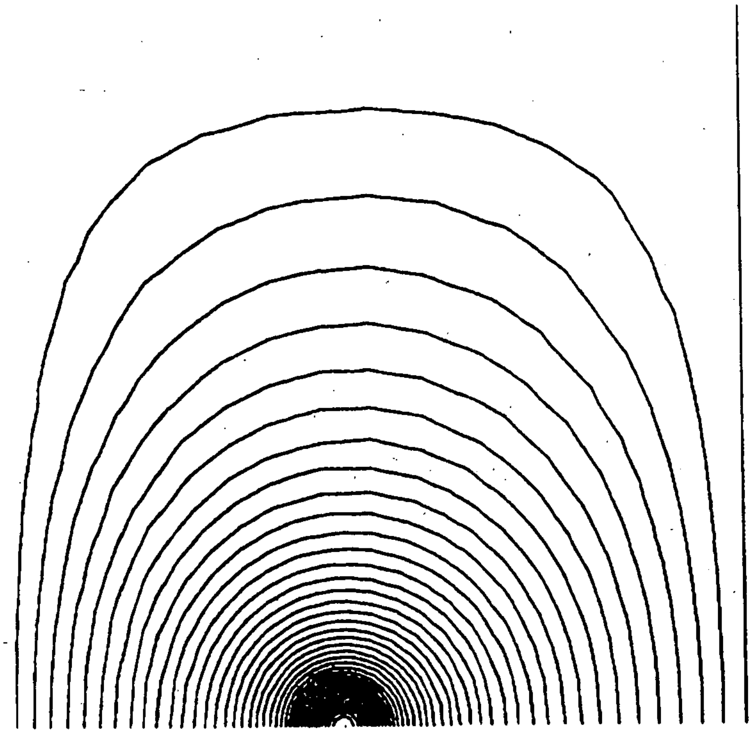


Fig. 35 : Streamlines $\psi \equiv u\text{-}Wr\text{-}k$ = constant and upper vorticity half-region $A_\psi^+ = \{x \in \Omega^+ ;\ x=(r,z),\ \psi(x) > 0$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.7) with λ=1., η=20. and W=.0001 is prescribed. The computed value of k is 6.1584.

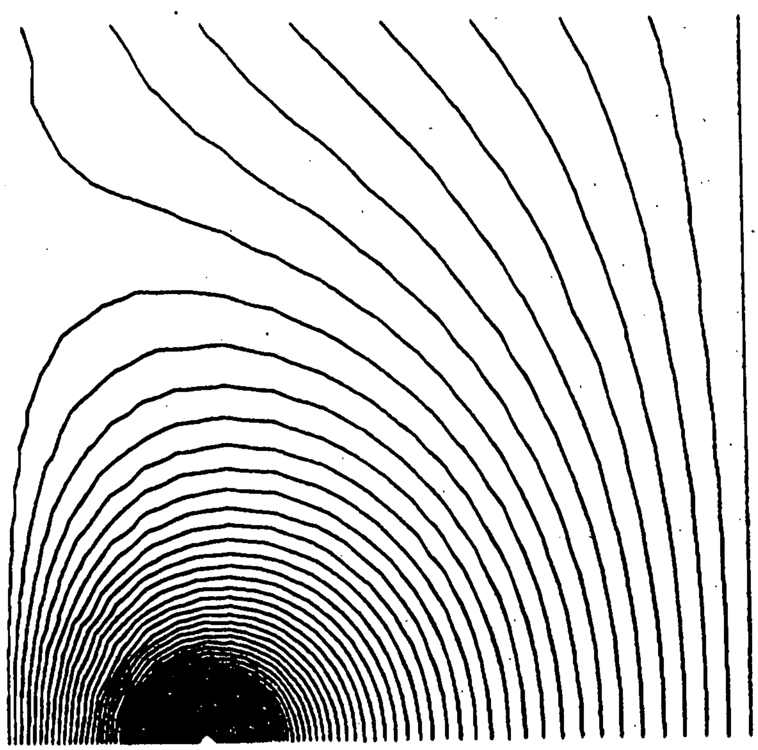


Fig. 36 : Streamlines $\psi \equiv u\text{-}Wr\text{-}k$ = constant and upper vorticity half-region $A_\psi^+ = \{x \in \Omega^+ ;\ x=(r,z),\ \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.7) with λ=1., η=20. and W=.001 is prescribed. The computed value of k is 5.8312.

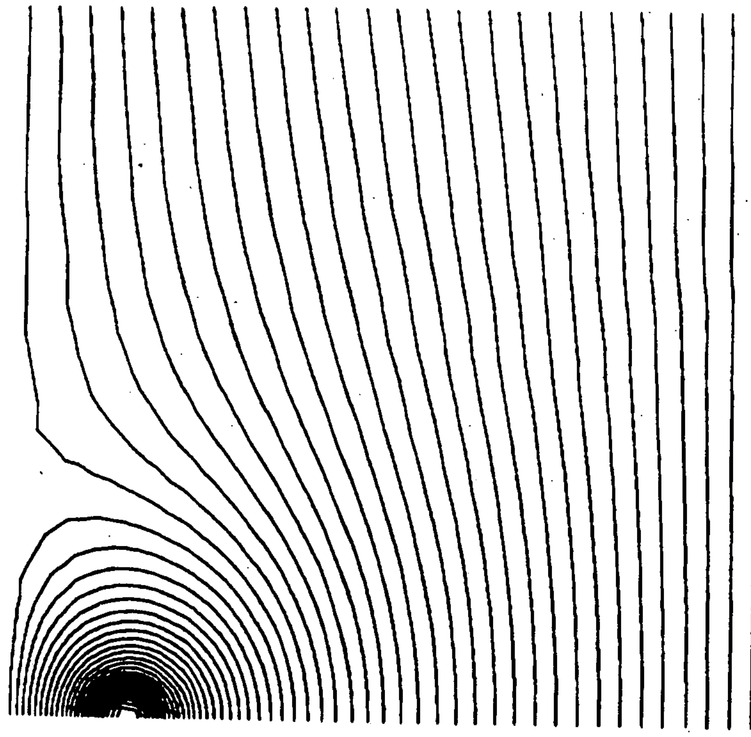


Fig. 37 : Streamlines $\psi \equiv u-Wr-k$ = constant and upper vorticity half-region $A_\psi^+ = \{x \in \Omega^+ \; ; \; x=(r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.7) with λ=1., η=20. and W=.01 is prescribed. The computed value of k is 4.2118.

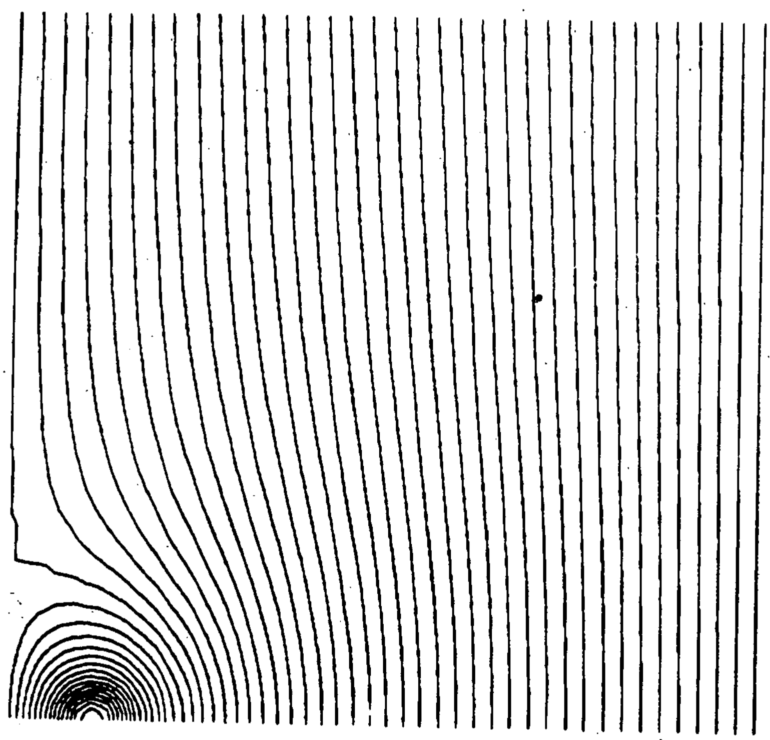


Fig. 38 : Streamlines $\psi \equiv u-Wr-k$ = constant and upper vorticity half-region $A_\psi^+ = \{x \in \Omega^+ \; ; \; x=(r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.7) with λ=1., η=20. and W=.1 is prescribed. The computed value of k is 3.5275.

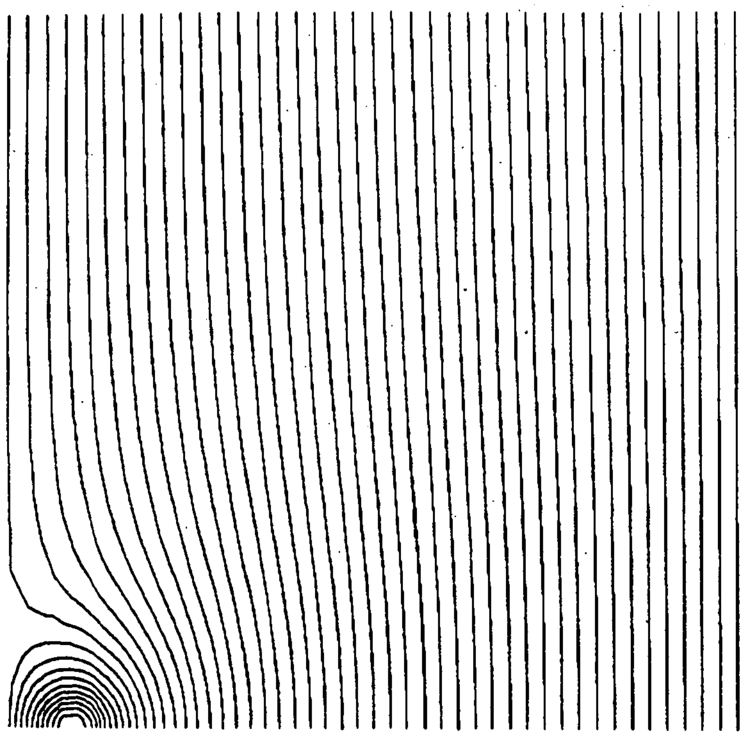


Fig. 39 : Streamlines $\psi \equiv u\text{-}Wr\text{-}k$ = constant and upper vorticity half-region $A_\psi^+ = \{x \in \Omega^+ \; ; \; x=(r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.7) with $\lambda=.1$, $\eta=20.$ and W=.2 is prescribed. The computed value of k is 3.0418.

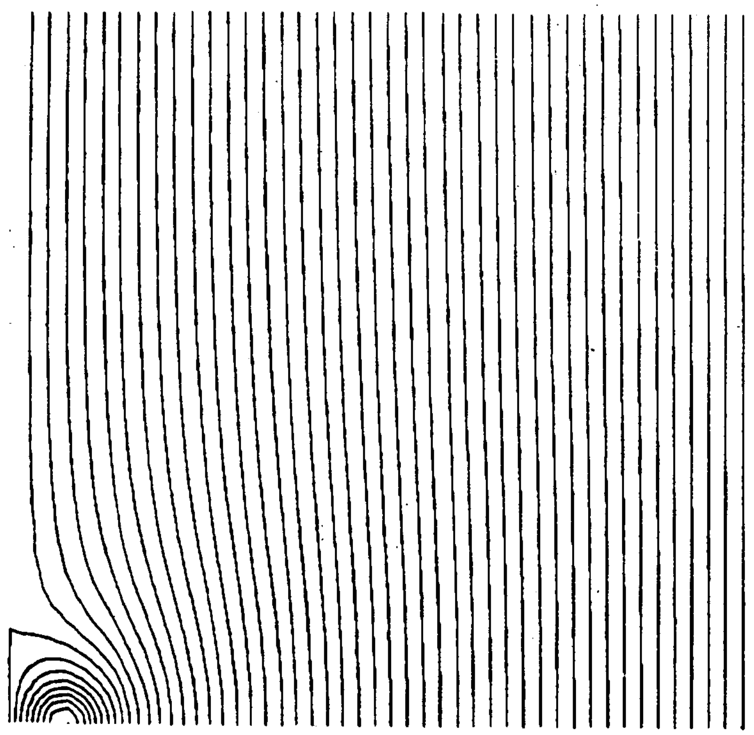


Fig. 40 : Streamlines $\psi \equiv u\text{-}Wr\text{-}k$ = constant and upper vorticity half-region $A_\psi^+ = \{x \in \Omega^+ \; ; \; x=(r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.7) with $\lambda=1.$, $\eta=20.$ and W=.3 is prescribed. The computed value of k is 2.1028.

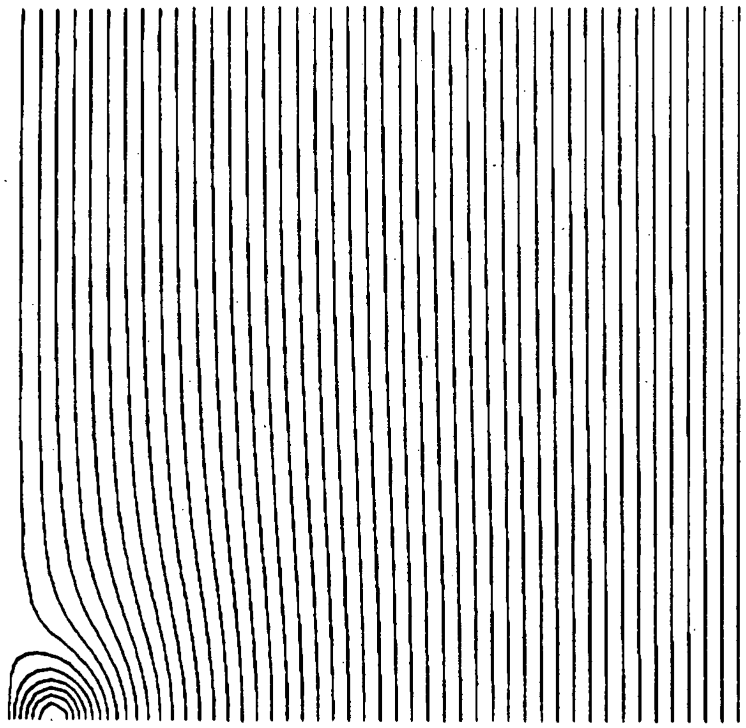


Fig. 41 : Streamlines $\psi \equiv u-Wr-k$ = constant and upper vorticity half-region $A_\psi^+ = \{x \in \Omega^+ \ ; \ x=(r,z), \ \psi(x) \geq 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.7) with λ=1., η=20. and W=.35 is prescribed. The computed value of k is 1.5047.

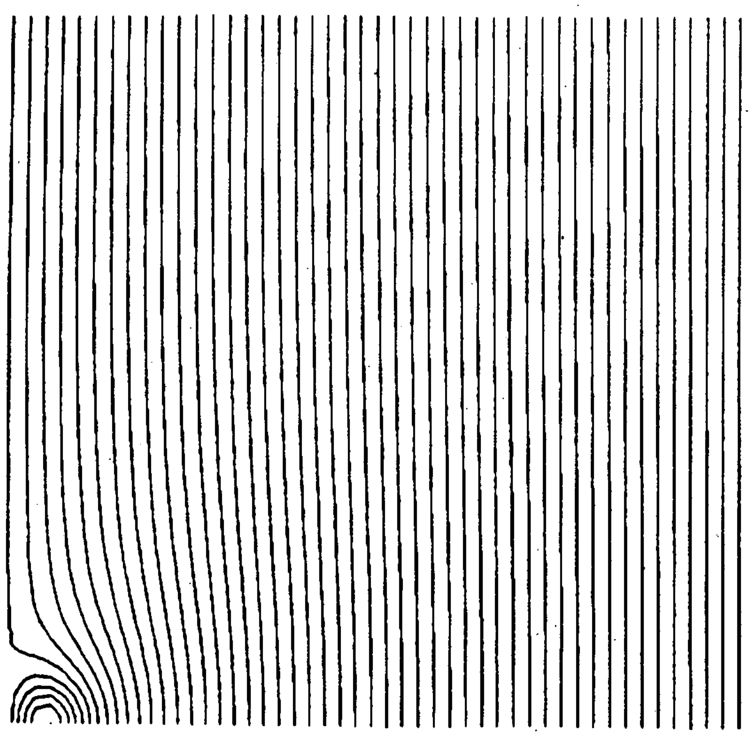


Fig. 42 : Streamlines $\psi \equiv u-Wr-k$ = constant and upper vorticity half-region $A_\psi^+ = \{x \in \Omega^+ \ ; \ x=(r,z), \ \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.7) with λ=1., η=20. and W=. is prescribed. The computed value of k is .9920.

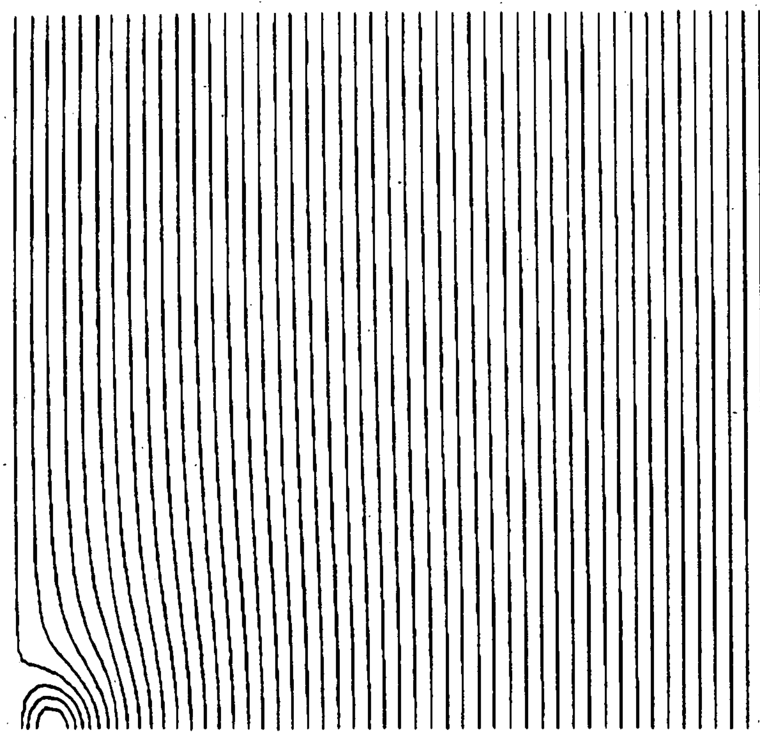


Fig. 43 : Streamlines $\psi \equiv u-Wr-k$ = constant and upper vorticity half-region $A_\psi^+ = \{x \in \Omega^+ \; ; \; x=(r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.7) with λ=1., η=20. and W=.45 is prescribed. The computed value of k is .1233.

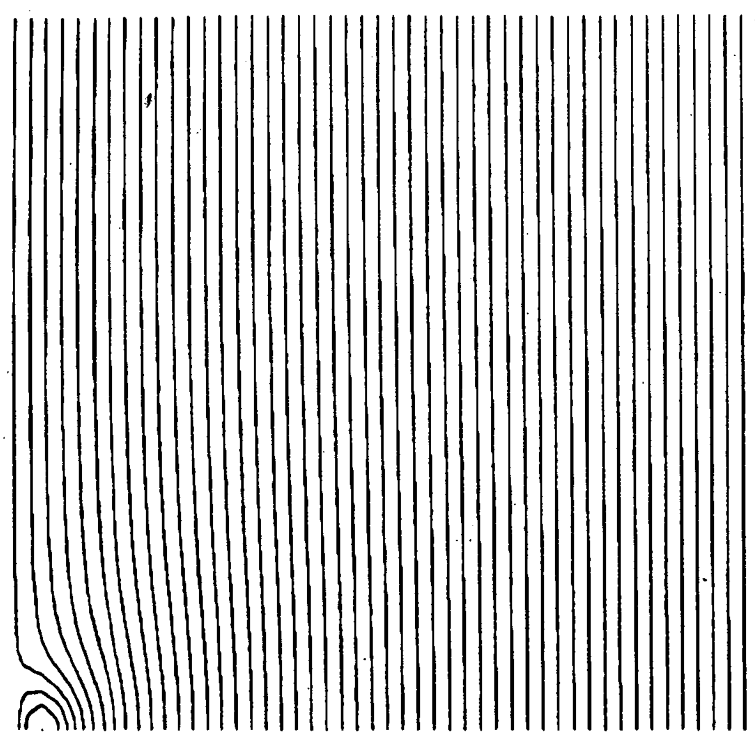


Fig. 44 : Streamlines $\psi \equiv u-Wr-k$ = constant and upper vorticity half-region $A_\psi^+ = \{x \in \Omega^+ \; ; \; x(r,z), \; \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.7) with λ=1., η=20. and W=.5 is prescribed. The computed value of k is -.0283.

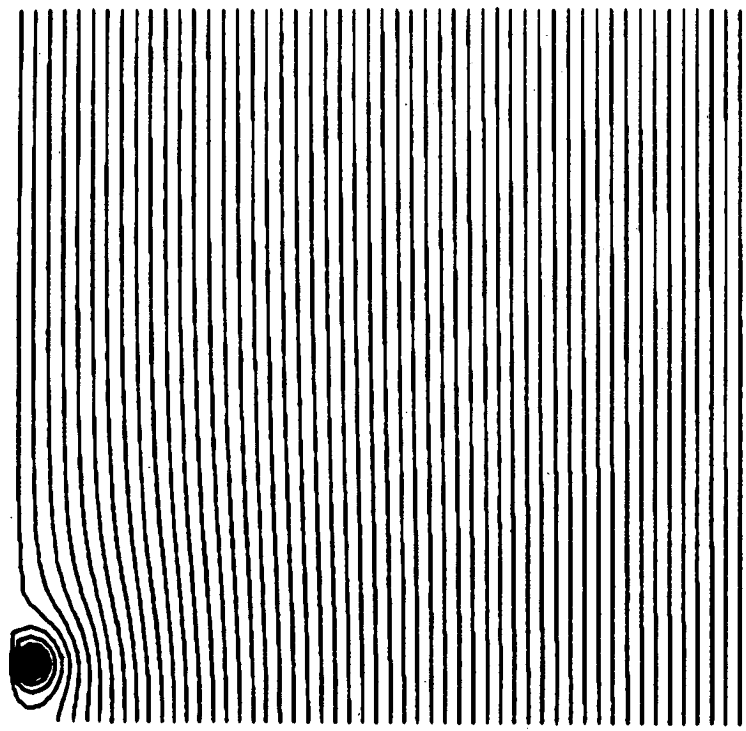


Fig. 45 : Streamlines $\psi \equiv u-Wr-k$ = constant and upper vorticity half-region $A_\psi^+ = \{x \in \Omega^+ ;\ x=(r,z),\ \psi(x) > 0\}$ corresponding to the computed solution of (10.1) where $\Omega = R(50.)$, f is given by (10.7) with λ=1., η=20. and W=.6 is prescribed. The computed value of k is -.0994.

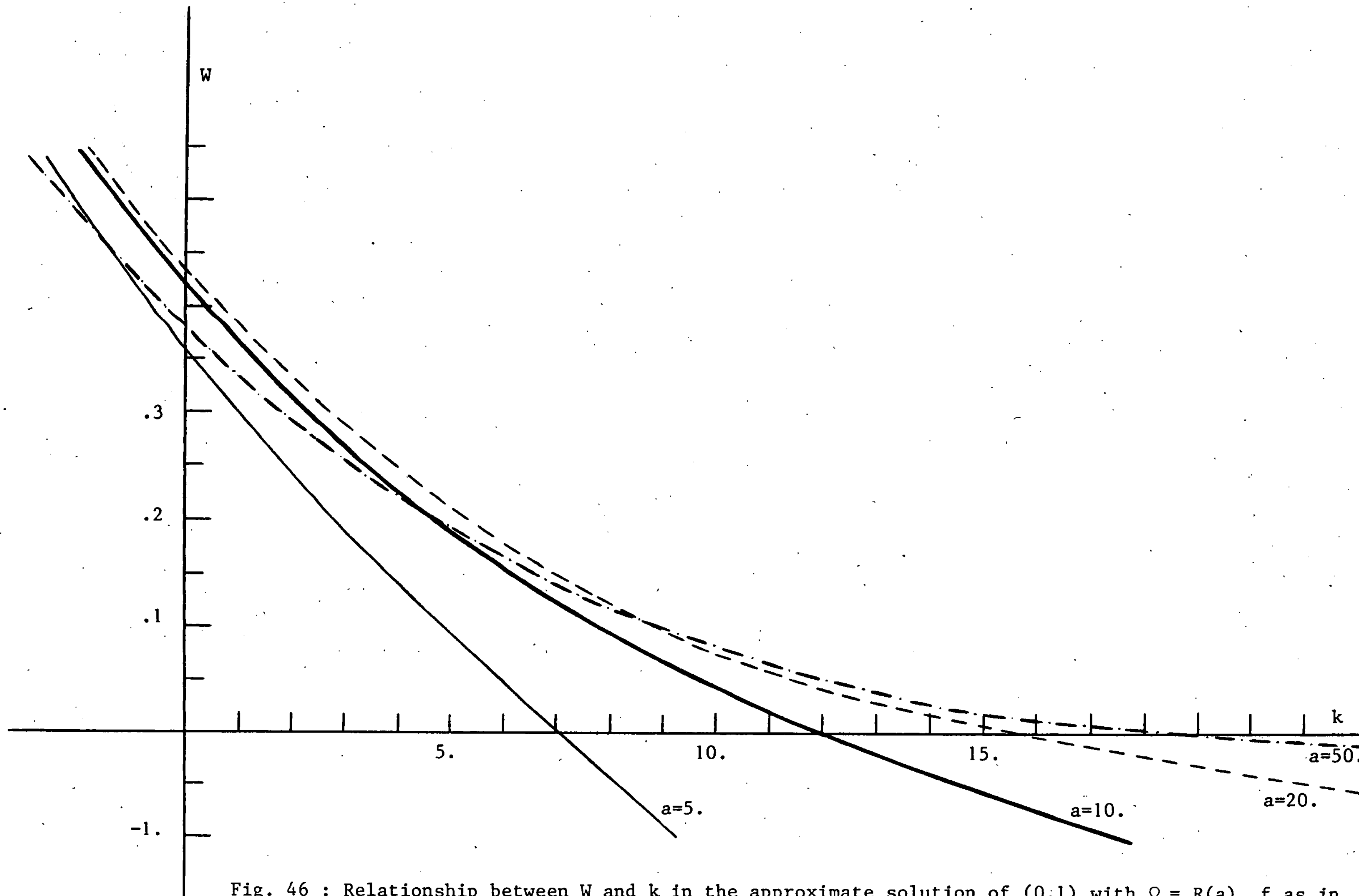


Fig. 46 : Relationship between W and k in the approximate solution of (0.1) with Ω = R(a), f as in (10.7) with λ=1. and η=1000. for different values of a.

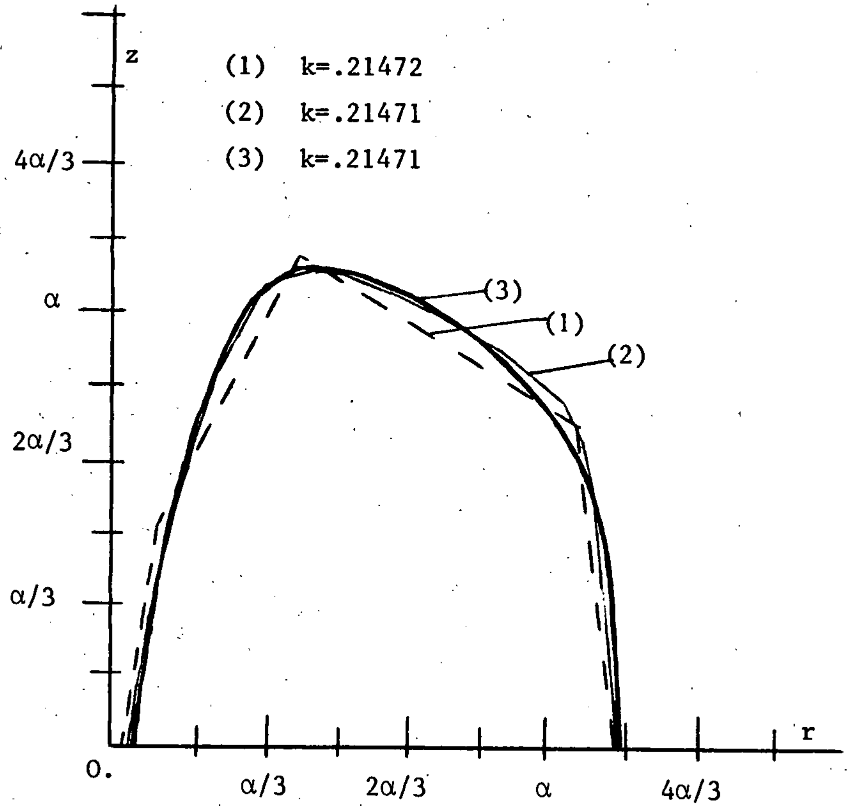


Fig. 47 : Upper-vorticity half-region corresponding to the computed solution of (10.1) with the data as in (10.25) and W=3.2.

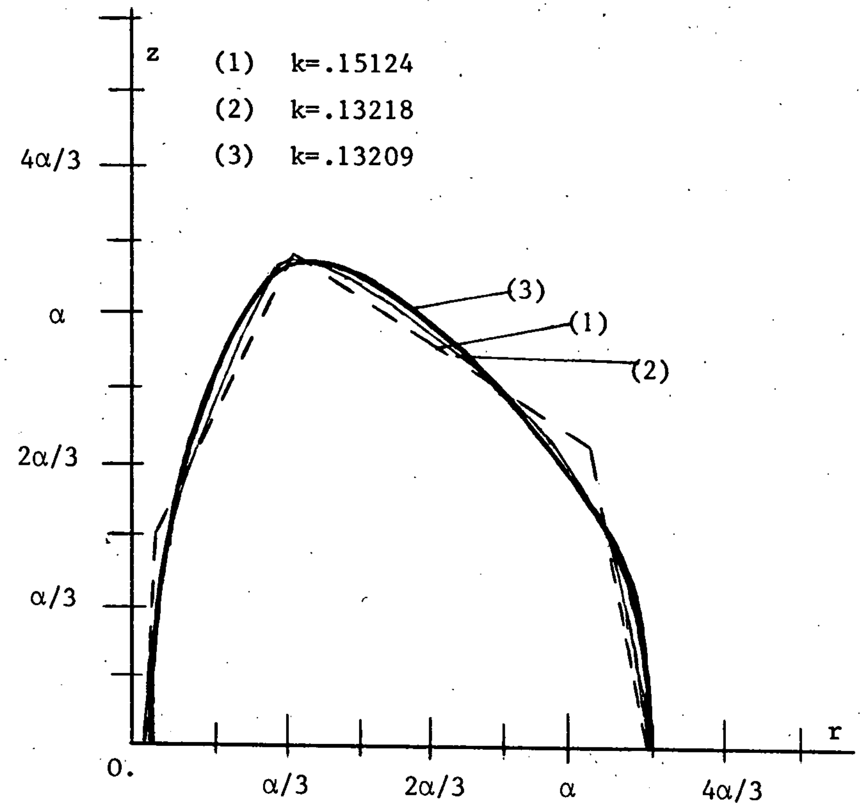


Fig. 48 : Upper vorticity half-region corresponding to the computed solution of (10.1) with the data as in (10.25) and W=3.225.

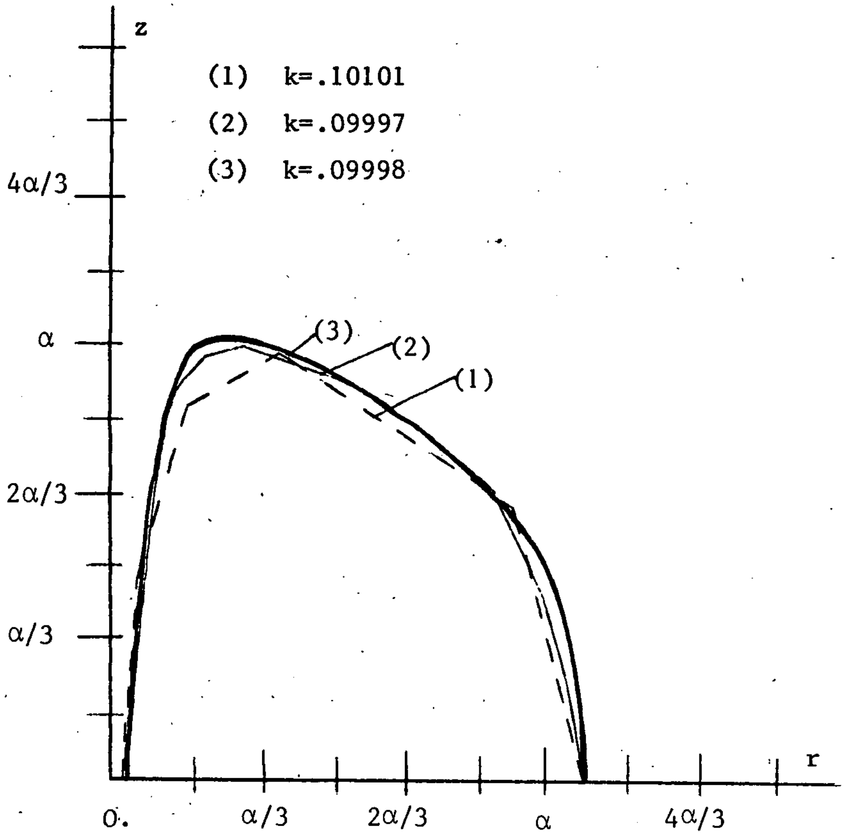


Fig. 49 : Upper vorticity half-region corresponding to the computed solution of (10.1) with the data as in (10.25) and W=3.25.

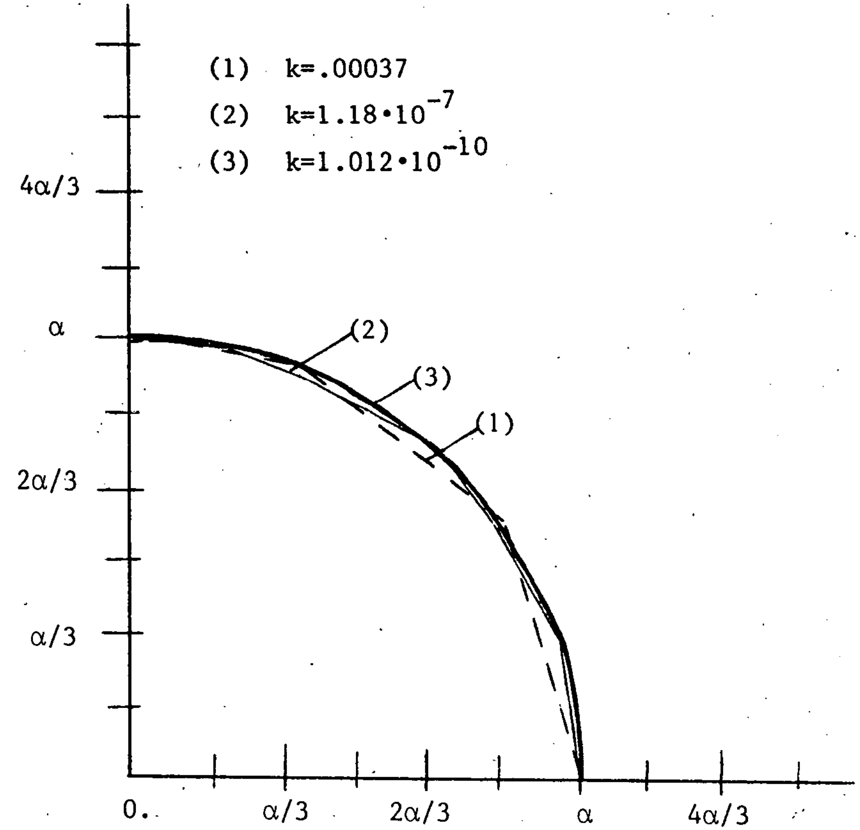


Fig. 50 : Upper vorticity half-region corresponding to the computed solution of (10.1) with the data as in (10.25) and W=3.29305.

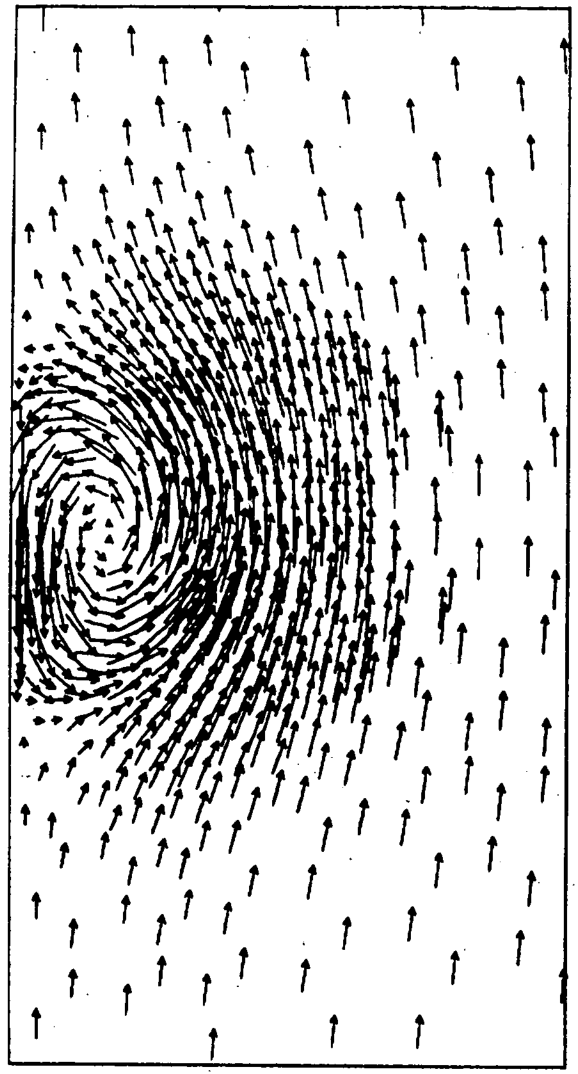


Fig. 51 : Detail of the velocity field corresponding to the computed solution of (10.1) with the data as in (10.25) and W = 3.29305. The field is displayed in the subdomain R(19.).

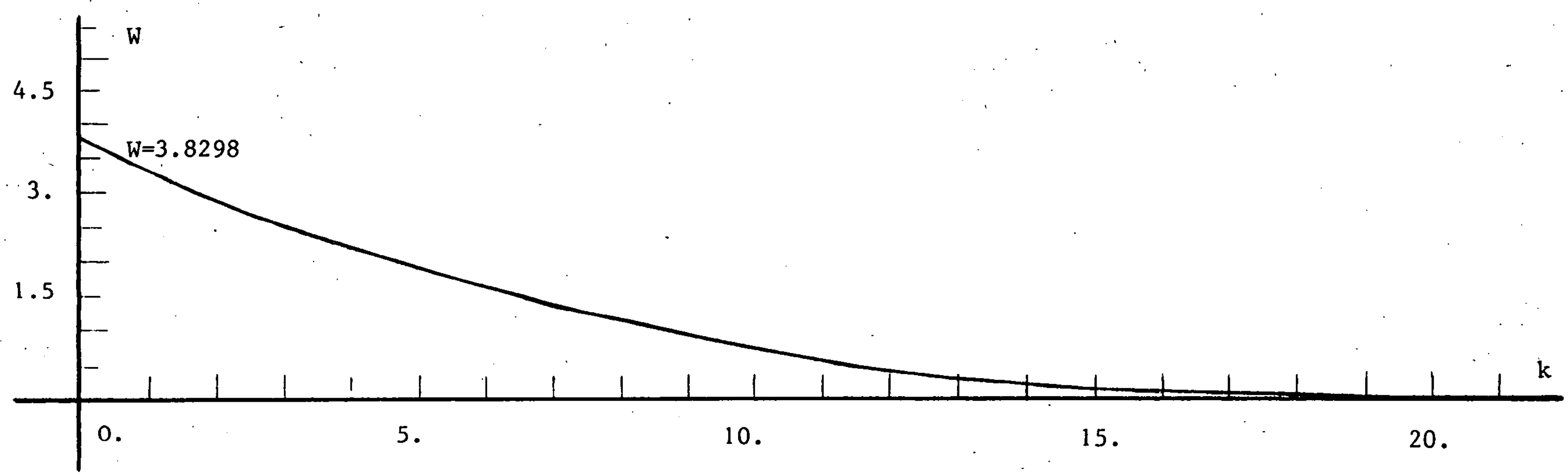
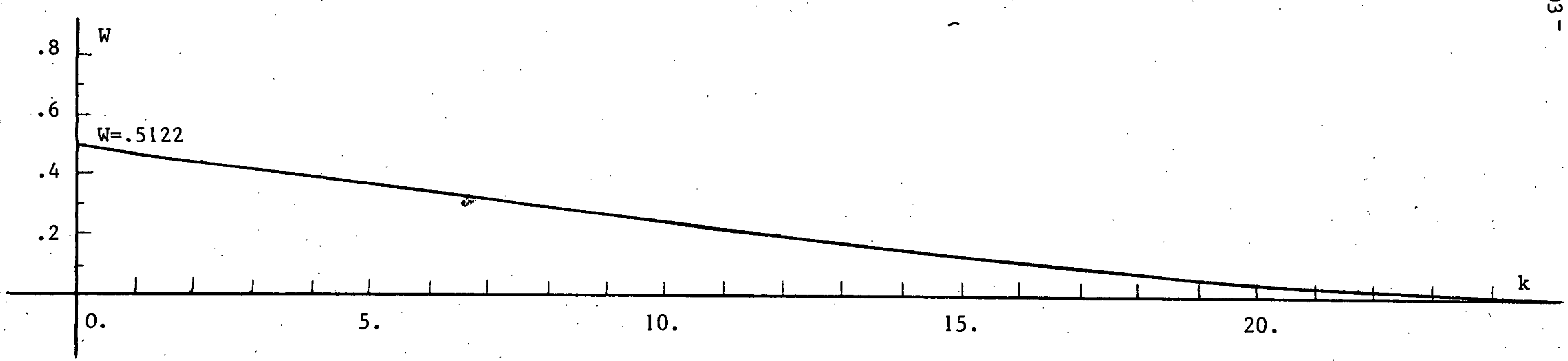


Fig. 52 : The asymptotic relationship between W and k for the two-dimensional vortex problem (10.1) where $\Omega = R(50.)$, f is given by (10.7) with $\lambda=2.$ and $\eta=1000$.



<u>Fig. 53</u> : The asymptotic relationship between W and k for the three-dimensional vortex problem (10.22) where Ω = B(100.), f is given by (10.23) with λ=2. and η=1000.

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